

# INVARIANT SPACE UNDER HÉNON RENORMALIZATION : INTRINSIC GEOMETRY OF CANTOR ATTRACTOR

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ABSTRACT. Three dimensional Hénon-like map

$$F(x, y, z) = (f(x) - \varepsilon(x, y, z), x, \delta(x, y, z))$$

is defined on the cubic box  $B$ . An invariant space under renormalization would appear only in higher dimension. Consider renormalizable maps each of which satisfies the condition

$$\partial_y \delta \circ F(x, y, z) + \partial_z \delta \circ F(x, y, z) \cdot \partial_x \delta(x, y, z) \equiv 0$$

for  $(x, y, z) \in B$ . Denote the set of maps satisfying the above condition be  $\mathcal{N}$ . Then the set  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$  is invariant under the renormalization operator where  $\mathcal{I}(\bar{\varepsilon})$  is the set of infinitely renormalizable maps. Hénon like diffeomorphism in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$  has universal numbers,  $b_2 \asymp |\partial_z \delta|$  and  $b_1 = b_F/b_2$  where  $b_F$  is the average Jacobian of  $F$ . The Cantor attractor of  $F \in \mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ ,  $\mathcal{O}_F$  has *unbounded geometry* almost everywhere in the parameter space of  $b_1$ . If two maps in  $\mathcal{N}$  has different universal numbers  $b_1$  and  $\tilde{b}_1$ , then the homeomorphism between two Cantor attractor is at most Hölder continuous, which is called *non rigidity*.

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## 1. Introduction

Hénon renormalization with universal limit was introduced in [dCLM]. Hénon-like map is

$$F(x, y) = (f(x) - \varepsilon(x, y), x)$$

two dimensional. The interesting maps in [dCLM] are Hénon-like with strong hyperbolic fixed points and its period doubling renormalization. In higher dimension, we are interested

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in the case that the invariant set has the only one repulsive or neutral direction and other directions is strongly contracting. Hénon renormalization was extended to three dimensional Hénon-like maps from a cubic box,  $B$  to itself

$$F(x, y, z) = (f(x) - \varepsilon(x, y, z), x, \delta(x, y, z))$$

in [Nam1]. In three dimension, the geometric properties of Cantor attractor were studied for the map in invariant spaces under renormalization operator rather than in the set of all infinitely renormalizable maps. The first invariant space is the set of toy model maps. In [Nam1], Hénon-like maps in this set has embedded invariant surfaces with the assumption of strong contraction along  $z$ -axis.

However, in this paper we discover another invariant space in which any Hénon-like map does not require any invariant surfaces. Instead of this, we see the formula between derivatives of the third coordinate map of  $F$ ,  $D\delta$  and  $D\delta_1$  of  $RF$  in Lemma A.1. Since renormalized map,  $RF$  is determined by  $F$ , the third coordinate map  $\delta_1$  of  $RF$  is so. In particular,  $f$ ,  $\varepsilon$  and  $\delta$  of  $F$  affects  $\delta_1$  in general. However, if  $F$  satisfies the following equation,

$$(1.1) \quad \partial_y \delta \circ (F(x, y, z)) + \partial_z \delta \circ (F(x, y, z)) \cdot \partial_x \delta(x, y, z) \equiv 0$$

then only  $D\delta$  affects the next third coordinate map,  $D\delta_1$ . Let  $\mathcal{N}$  be the set of renormalizable Hénon-like maps each of which satisfies the above equation (1.1). Then the set  $\mathcal{I}(\bar{\varepsilon}) \cap \mathcal{N}$  is invariant under renormalization operator (Theorem 3.5). Moreover, there exist two universal numbers, say  $b_1$  and  $b_2$  and  $b_1 b_2 = b_F$  by Proposition 4.5 and Lemma 4.5. Each of three dimensional Hénon-like maps in this space has its Cantor attractor which has *unbounded geometry* and *non-rigidity* (Theorem 6.7 and Theorem 7.2). It is worth to emphasize that these two geometric properties of Cantor attractor only depend on  $b_1$  which is from the two dimensional Hénon-like map in three dimension.

## 2. Preliminaries

Let three dimensional *Hénon-like map*  $F$  be the map as follows

$$F(x, y, z) = (f(x) - \varepsilon(x, y, z), x, \delta(x, y, z))$$

where  $f$  is the unimodal map and  $\varepsilon$  and  $\delta$  from  $\text{Dom}(F)$  to  $\mathbb{R}$  are maps with small norms, that is,  $\|\varepsilon\|, \|\delta\| \leq \bar{\varepsilon}$  for small enough  $\bar{\varepsilon} > 0$ . Then the image of the plane  $\{x = \text{const.}\}$  under Hénon-like map is the plane  $\{y = \text{const.}\}$ .

Let  $f$  be the unimodal map defined on the closed interval  $I$  such that  $f(I) \subset I$  and the critical point and the critical value are in  $I$ .  $f$  is *renormalizable* with period doubling type if there exists a closed subinterval  $J \neq I$  containing the critical point,  $c_f$  of  $f$  and  $f^2(J)$  is invariant and  $f^2(c_f) \in \partial J$ . Thus if  $f$  is renormalizable, then we can consider the smallest interval  $J_f$  satisfying the above properties. The conjugation of the appropriate affine rescaling of  $f^2|_{J_f}$  defines the renormalization of  $f$ ,  $Rf: I \rightarrow I$ . If  $f$  is infinitely renormalizable, there exists the renormalization fixed point  $f_*$ . The *scaling factor* of  $f_*$  is

$$\sigma = \frac{|J_{f_*}|}{|I|}$$

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and  $\lambda = 1/\sigma = 2.6 \dots$ . For the properties of renormalizable unimodal maps on the bounded interval, for example, see [BB]. However, the image of  $\{x = \text{const.}\}$  under  $F^2$  is the surface

$$f(x) - \varepsilon(x, y, z) = \text{const.}$$

which is not a plane except that  $\varepsilon \equiv 0$ . Thus analytic definition of renormalization of  $F$  requires non-linear coordinate change map. The *horizontal-like* diffeomorphism  $H$  of  $F$  is defined as follows

$$H(x, y, z) = (f(x) - \varepsilon(x, y, z), y, z - \delta(y, f^{-1}(y), 0))$$

and it preserves the plane  $\{y = \text{const.}\}$ . Then *renormalization* of  $F$  is defined

$$RF = \Lambda \circ H \circ F^2 \circ H^{-1} \circ \Lambda^{-1}$$

where  $\Lambda(x, y, z) = (sx, sy, sz)$  for the appropriate constant  $s < -1$ . If  $F$  is  $n$ -times renormalizable, then  $R^k F$  is defined as the renormalization of  $R^{k-1} F$  for  $2 \leq k \leq n$ . Denote  $\text{Dom}(F)$  to be the cubic box region,  $B$ . If the set  $B$  is emphasized with the relation of a certain map  $R^k F$ , for example, then denote this region to be  $B(R^k F)$ .

Let  $\mathcal{I}(\bar{\varepsilon})$  be the space of the infinitely renormalizable Hénon-like maps with small norm,  $\|\varepsilon\|, \|\delta\| = O(\bar{\varepsilon})$ . If  $\bar{\varepsilon} > 0$  is sufficiently small, then renormalization operator  $R$  has the unique fixed point in  $\mathcal{I}(\bar{\varepsilon})$ . The fixed point  $F_*$  is the degenerate map which is of following form

$$F(x, y, z) = (f_*(x), x, 0).$$

$F_*$  is the hyperbolic fixed point of the renormalization operator and it has codimension one stable manifold at  $F_*$ .

$F_k$  denotes  $R^k F$  for each  $k$ . Let the coordinate change map which conjugates  $F_k^2|_{\Lambda_k^{-1}(B)}$  and  $RF_k$  is denoted by

$$\psi_v^{k+1} \equiv H_k^{-1} \circ \Lambda_k^{-1}: \text{Dom}(RF_k) \rightarrow \Lambda_k^{-1}(B)$$

where  $H_k$  is the horizontal-like diffeomorphism and  $\Lambda_k$  is dilation with the constant  $s_k < -1$ . Let  $\psi_c^{k+1} = F_k \circ \psi_v^{k+1}$ . For  $k < n$  and express the compositions of  $\psi_v^{k+1}$  and  $\psi_c^{k+1}$  as follows

$$\begin{aligned} \Psi_{k, \mathbf{v}}^n &= \psi_v^{k+1} \circ \psi_v^{k+2} \circ \dots \circ \psi_v^n \\ \Psi_{k, \mathbf{c}}^n &= \psi_c^{k+1} \circ \psi_c^{k+2} \circ \dots \circ \psi_c^n. \end{aligned}$$

Moreover, the word of length  $n$  in the Cartesian product,  $\{v, c\}^n$  be  $\mathbf{w}_n$  or simply  $\mathbf{w}$ . The map  $\Psi_{k, \mathbf{w}}^n$  is from  $B(R^n F)$  to  $B(R^k F)$  where the word  $\mathbf{w}$  of the length  $n - k$ . Denote the region  $\Psi_{k, \mathbf{w}}^n(B(R^n F))$  by  $B_{\mathbf{w}}^n$ . We see that  $\text{diam}(B_{\mathbf{v}}^n) \leq C\sigma^n$  where  $\mathbf{v} = v^n$  for some  $C > 0$  in [dCLM] or [Nam1]. If  $F$  is a infinitely renormalizable Hénon-like map, then it has invariant Cantor set

$$\mathcal{O}_F = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{w} \in W^n} B_{\mathbf{w}}^n$$

and  $F$  acts on  $\mathcal{O}_F$  as a dyadic adding machine. The counterpart of the critical value of the unimodal renormalizable map is called the *tip*

$$\{\tau_F\} \equiv \bigcap_{n \geq 0} B_{v^n}^n.$$

The unique invariant probability measure on  $\mathcal{O}_F$  is denoted by  $\mu$ . The average Jacobian of  $F$ ,  $b_F$  is defined as

$$b_F = \exp \int_{\mathcal{O}_F} \log \text{Jac } F \, d\mu.$$

Then there exists the asymptotic expression of  $\text{Jac } R^n F$  for the map  $F \in \mathcal{I}(\bar{\varepsilon})$  with  $b_F$  and the universal function.

**Theorem 2.1** ([Nam2]). *For the map  $F \in \mathcal{I}(\bar{\varepsilon})$  with small enough positive number  $\bar{\varepsilon}$ , the Jacobian determinant of renormalization of  $F$  as follows*

$$(2.1) \quad \text{Jac } R^n F = b_F^{2^n} a(x) (1 + O(\rho^n))$$

where  $b_F$  is the average Jacobian of  $F$  and  $a(x)$  is the universal positive function for  $n \in \mathbb{N}$  and for some  $\rho \in (0, 1)$ .

Denote the tip,  $\tau_{F_n}$  for  $n \in \mathbb{N}$  to be  $\tau_n$ . The definitions of tip and  $\Psi_{k, \mathbf{v}}^n$  imply that  $\Psi_{k, \mathbf{v}}^n(\tau_n) = \tau_k$  for  $k < n$ . Then after composing appropriate translations, tips on each level moves to the origin as the fixed point

$$\Psi_k^n(w) = \Psi_{k, \mathbf{v}}^n(w + \tau_n) - \tau_k$$

for  $k < n$ . The map  $\Psi_k^n$  is separated non linear part and dilation part after reshuffling

$$\Psi_k^n(w) = \begin{pmatrix} 1 & t_{n,k} & u_{n,k} \\ & 1 & \\ d_{n,k} & & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,k} & & \\ & \sigma_{n,k} & \\ & & \sigma_{n,k} \end{pmatrix} \begin{pmatrix} x + S_k^n(w) \\ y \\ z + R_k^n(y) \end{pmatrix}$$

where  $\alpha_{n,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$  and  $\sigma_{n,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$ . In this paper, we confuse the map  $\Psi_{k, \mathbf{v}}^n$  with  $\Psi_k^n$  to obtain the simpler expression of each coordinate map of  $\Psi_{k, \mathbf{v}}^n$ . For example, the third coordinate expression of  $\Psi_k^n$

$$\sigma_{n,k} d_{n,k} y + \sigma_{n,k} [z + R_k^n(y)]$$

means that  $\sigma_{n,k} d_{n,k} (y + \tau_n^y) + \sigma_{n,k} [z + \tau_n^z + R_k^n(y + \tau_n^y)] - \tau_k$  where  $\tau_n = (\tau_n^x, \tau_n^y, \tau_n^z)$ . By the same way, the first coordinate map

$$\alpha_{n,k} [x + S_k^n(w)] + \sigma_{n,k} t_{n,k} y + \sigma_{n,k} u_{n,k} [z + R_k^n(y)]$$

means that

$$\alpha_{n,k} [(x + \tau_n^x) + S_k^n(w + \tau_n)] + \sigma_{n,k} t_{n,k} (y + \tau_n^y) + \sigma_{n,k} u_{n,k} [(z + \tau_n^z) + R_k^n(y + \tau_n^y)] - \tau_k$$

for  $k < n$ . Recall the definitions for later use

$$\begin{aligned} \Lambda_n^{-1}(w) &= \sigma_n \cdot w, & \psi_v^{n+1}(w) &= H_n^{-1}(\sigma_n w), & \psi_c^{n+1}(w) &= F_n \circ H_n^{-1}(\sigma_n w) \\ \psi_v^{n+1}(B(R^{n+1}F)) &= B_v^{n+1}, & \psi_c^{n+1}(B(R^{n+1}F)) &= B_c^{n+1} \end{aligned}$$

for each  $n \in \mathbb{N}$ .

### 3. An invariant space under renormalization

**3.1. A space of renormalizable maps from recursive formulas of  $D\delta$ .** Let  $F$  be a renormalizable three dimensional Hénon-like map. Denote partial derivatives of the composition as follows

$$\partial_x\{P \circ Q(w)\} \equiv \partial_x P(Q(w)) \quad \partial_x P \text{ at } Q(w) \text{ is } \partial_x P \circ Q(w).$$

The similar notation is defined for partial derivatives over any other variables also. Recall that  $RF$  is the renormalized map of  $F$  and its third coordinate map of  $RF$  is  $\delta_1 = \pi_z \circ RF$ . Thus by the definition of renormalization, we obtain the relation between  $\delta$  and  $\delta_1$  as follows

$$\delta_1(w) = \sigma_0 \cdot [\delta \circ F \circ H^{-1}(\sigma_0 w) - \delta(\sigma_0 x, f^{-1}(\sigma_0 x), 0)].$$

Then by Lemma A.1, we obtain

$$\begin{aligned} \partial_x \delta_1(w) &= \boxed{[\partial_y \delta \circ \psi_c^1(w) + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_x \delta \circ \psi_v^1(w)]} \cdot \partial_x \phi^{-1}(\sigma_0 w) \\ &\quad + \partial_x \delta \circ \psi_c^1(w) - \frac{d}{dx} \delta(\sigma_0 x, f^{-1}(\sigma_0 x), 0) \\ \partial_y \delta_1(w) &= \boxed{[\partial_y \delta \circ \psi_c^1(w) + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_x \delta \circ \psi_v^1(w)]} \cdot \partial_y \phi^{-1}(\sigma_0 w) \\ &\quad + \partial_z \delta \circ \psi_c^1(w) \cdot \left[ \partial_y \delta \circ \psi_v^1(w) + \partial_z \delta \circ \psi_v^1(w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right] \\ \partial_z \delta_1(w) &= \boxed{[\partial_y \delta \circ \psi_c^1(w) + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_x \delta \circ \psi_v^1(w)]} \cdot \partial_z \phi^{-1}(\sigma_0 w) \\ &\quad + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_z \delta \circ \psi_v^1(w) \end{aligned}$$

Then we can consider the maps such that above boxed expression is identically zero.

**Definition 3.1.** Let  $\mathcal{N}$  be the set of renormalizable three dimensional Hénon-like maps each of which satisfies the following equation

$$\partial_y \delta \circ (F(w)) + \partial_z \delta \circ (F(w)) \cdot \partial_x \delta(w) = 0$$

for all  $w \in \psi_v^1(B) \cup \psi_c^1(B)$ .

**Example 3.1.** Hénon-like maps in the above class  $\mathcal{N}$  is non empty and non trivial. For instance, suppose that the third coordinate map of the given Hénon-like map  $F$  is

$$\delta(x, y, z) = \eta(Cy - z) + Cx$$

where  $\max\{\|\eta\|_{C^3}, |C|\} = O(\bar{\varepsilon})$  for  $C \in \mathbb{R}$ . Then  $F$  is in  $\mathcal{N}$ . The map  $\eta$  can be arbitrary with small norm.

In the rest of this paper, we use the notation  $q(y)$  and  $q_k(y)$  as follows

$$(3.1) \quad q(y) = \frac{d}{dy} \delta(y, f^{-1}(y), 0), \quad q_k(y) = \frac{d}{dy} \delta_k(y, f_k^{-1}(y), 0)$$

for each  $k \in \mathbb{N}$ . Moreover, the value of  $q_k$  at  $\sigma_k y$  is expressed as  $q_k \circ (\sigma_k y)$ .

### 3.2. Invariance of the space $\mathcal{N}$ under renormalization.

**Lemma 3.2.** *Suppose that Hénon-like map  $F_{k-1}$  is renormalizable. Denote  $RF_{k-1}$  by  $F_k$  and  $\pi_z \circ F_j(x, y, z)$  by  $\delta_j(x, y, z)$  for  $j = k-1, k$ . Then*

$$\begin{aligned}\partial_x \delta_k(w) &= \partial_x \delta_{k-1} \circ \psi_c^k(w) - q_{k-1} \circ (\sigma_{k-1} \cdot x) \\ \partial_y \delta_k(w) &= \partial_z \delta_{k-1} \circ \psi_c^k(w) \cdot [\partial_y \delta_{k-1} \circ \psi_v^k(w) + \partial_z \delta_{k-1} \circ \psi_v^k(w) \cdot q_{k-1}(\sigma_{k-1} \cdot y)] \\ \partial_z \delta_k(w) &= \partial_z \delta_{k-1} \circ \psi_c^k(w) \cdot \partial_z \delta_{k-1} \circ \psi_v^k(w).\end{aligned}$$

*Proof.* See Lemma A.1 and use the induction.  $\square$

**Lemma 3.3.** *Let  $F$  be an infinitely renormalizable Hénon-like map and let  $F_k$  be  $R^k F$  for each  $k \in \mathbb{N}$ . Then*

$$\begin{aligned}\psi_v^k \circ F_k &= F_{k-1} \circ \psi_c^k \\ \pi_y \circ \psi_v^k \circ F_k &= \pi_x \circ \psi_c^k\end{aligned}$$

for each  $k \in \mathbb{N}$ .

*Proof.* Recall that  $\psi_v^k = H_{k-1} \circ \Lambda_{k-1}$ ,  $\psi_c^k = F_{k-1} \circ \psi_v^k$  and  $F_k = (\psi_v^k)^{-1} \circ F_{k-1}^2 \circ \psi_v^k$ . Then

$$\begin{aligned}(3.2) \quad \psi_v^k \circ F_k &= \psi_v^k \circ (\psi_v^k)^{-1} \circ F_{k-1}^2 \circ \psi_v^k \\ &= F_{k-1}^2 \circ \psi_v^k = F_{k-1} \circ [F_{k-1} \circ \psi_v^k] \\ &= F_{k-1} \circ \psi_c^k\end{aligned}$$

for  $k \in \mathbb{N}$ . The special form of Hénon-like map implies that  $\pi_y(F_{k-1}(w)) = \pi_x(w)$ . Hence, the equation (3.2) implies that  $\pi_y \circ \psi_v^k \circ F_k = \pi_x \circ \psi_c^k$ .  $\square$

Let us express the notation of  $\psi_w^k \circ \dots \circ \psi_w^n$  where  $w = v$  or  $c \in W$  as follows <sup>1</sup>

$$\begin{aligned}\psi_v^k \circ \psi_v^{k+1} \circ \dots \circ \psi_v^n &= \Psi_{k, v^{n-k}}^n \equiv \Psi_{k, \mathbf{v}}^n \\ \psi_c^k \circ \psi_c^{k+1} \circ \dots \circ \psi_c^n &= \Psi_{k, c^{n-k}}^n \equiv \Psi_{k, \mathbf{c}}^n\end{aligned}$$

Moreover, let us take the following notations

$$\Psi_{k, \mathbf{v}}^n \circ \psi_c^{n+1} \equiv \Psi_{k, \mathbf{vc}}^{n+1}, \quad \Psi_{k, \mathbf{c}}^n \circ \psi_v^{n+1} \equiv \Psi_{k, \mathbf{cv}}^{n+1}$$

for each  $n \in \mathbb{N}$ . The notation  $\Psi_{k, \mathbf{vcv}}^{n+2}$  or  $\Psi_{k, \mathbf{vc}^2}^{n+2}$  and any similar ones are allowed.

**Corollary 3.4.** *Let  $F$  be the infinitely renormalizable Hénon-like map. Then*

$$\Psi_{k, \mathbf{v}}^n \circ F_n = F_k \circ \Psi_{k, \mathbf{c}}^n$$

for  $k < n$ .

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<sup>1</sup>By the above definition,  $\Psi_{v^n}^n$  and  $\Psi_{c^n}^n$  can be also expressed as follows

$$\Psi_{v^n}^n = \Psi_{0, \mathbf{v}}^n, \quad \Psi_{c^n}^n = \Psi_{0, \mathbf{c}}^n$$

*Proof.* Recall the equation  $\psi_c^{j+1} = F_j \circ \psi_v^{j+1}$  for  $k \leq j < n$ . Thus

$$\begin{aligned}
F_k \circ \Psi_{k,\mathbf{c}}^n &= F_k \circ \psi_c^{k+1} \circ \psi_c^{k+2} \circ \dots \circ \psi_c^n \\
&= [F_k \circ (F_k \circ \psi_v^{k+1})] \circ (F_{k+1} \circ \psi_v^{k+2}) \circ \dots \circ (F_{n-1} \circ \psi_v^n) \\
&= \psi_v^{k+1} \circ \psi_v^{k+2} \circ F_{k+2} \circ (F_{k+2} \circ \psi_v^{k+3}) \circ \dots \circ (F_{n-1} \circ \psi_v^n) \\
&\quad \vdots \\
&= \psi_v^{k+1} \circ \psi_v^{k+2} \circ \dots \circ \psi_v^n \circ F_n \\
&= \Psi_{k,\mathbf{v}}^n \circ F_n
\end{aligned}$$

□

**Theorem 3.5.** *Let  $\mathcal{N}$  be the set of renormalizable Hénon-like maps defined in Definition 3.1. The space  $\mathcal{I}(\bar{\varepsilon}) \cap \mathcal{N}$  is invariant under renormalization, that is, if  $F \in \mathcal{I}(\bar{\varepsilon}) \cap \mathcal{N}$ , then  $RF \in \mathcal{I}(\bar{\varepsilon}) \cap \mathcal{N}$ .*

*Proof.* Recall that  $\pi_z \circ R^k F$  is  $\delta_k$  and  $D\delta_k(w) = (\partial_x \delta(w) \quad \partial_y \delta_k(w) \quad \partial_z \delta_k(w))$  for  $k \in \mathbb{N}$ . Suppose that

$$\partial_y \delta_{k-1} \circ (F_{k-1}(w)) + \partial_z \delta_{k-1} \circ (F_{k-1}(w)) \cdot \partial_x \delta_{k-1}(w) = 0$$

where  $w \in \psi_c^k(B) \cup \psi_v^k(B)$ . By induction it is sufficient to show that

$$\partial_y \delta_k \circ (F_k(w)) + \partial_z \delta_k \circ (F_k(w)) \cdot \partial_x \delta_k(w) = 0$$

where  $w \in \psi_c^{k+1}(B) \cup \psi_v^{k+1}(B)$ . Observe that  $\sigma_{k-1}y = \pi_y \circ \psi_v^k(w)$  and  $\sigma_{k-1}x = \pi_x \circ \psi_c^k(w)$ . By Lemma 3.3, we have that  $\pi_x \circ \psi_c^k(w) = \pi_y \circ \psi_v^k \circ F_k(w)$  and  $F_k \circ \psi_c^k(w) = \psi_v^k \circ F_k(w)$ . Then by Lemma 3.2,

$$\begin{aligned}
&\partial_y \delta_k \circ (F_k(w)) + \partial_z \delta_k \circ (F_k(w)) \cdot \partial_x \delta_k(w) \\
&= \partial_z \delta_{k-1} \circ \psi_c^k \circ F_k(w) \\
&\quad \cdot [\partial_y \delta_{k-1} \circ \psi_v^k \circ F_k(w) + \partial_z \delta_{k-1} \circ \psi_v^k \circ F_k(w) \cdot q_{k-1} \circ \pi_y \circ \psi_v^k \circ F_k(w)] \\
&\quad + \partial_z \delta_{k-1} \circ \psi_c^k \circ F_k(w) \cdot \partial_z \delta_{k-1} \circ \psi_v^k \circ F_k(w) \cdot [\partial_x \delta_{k-1} \circ \psi_c^k(w) - q_{k-1} \circ \pi_x \circ \psi_c^k(w)] \\
&= \partial_z \delta_{k-1} \circ \psi_c^k \circ F_k(w) \cdot [\partial_y \delta_{k-1} \circ \psi_v^k \circ F_k(w) \\
&\quad + \partial_z \delta_{k-1} \circ \psi_c^k \circ F_k(w) \cdot \partial_z \delta_{k-1} \circ \psi_v^k \circ F_k(w) \cdot \partial_x \delta_{k-1} \circ \psi_c^k(w)] \\
&= \partial_z \delta_{k-1} \circ \psi_c^k \circ F_k(w) \\
&\quad \cdot [\partial_y \delta_{k-1} \circ \psi_v^k \circ F_k(w) + \partial_z \delta_{k-1} \circ \psi_v^k \circ F_k(w) \cdot \partial_x \delta_{k-1} \circ \psi_c^k(w)] \\
&= 0
\end{aligned}$$

For any point  $w \in \text{Dom}(R^k F)$ , we obtain that  $\psi_c^k(w) \in \psi_c^k(B) \cup \psi_v^k(B)$ . Then  $RF_{k-1} \in \mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Hence, the space  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$  is invariant under renormalization. □

## 4. Universal numbers with $\partial_z \delta$ and $\partial_y \varepsilon$

**4.1. Critical point and recursive formula of  $\partial_x \delta_n$ .** Let us define the *critical point*,  $c_F$  of  $F \in \mathcal{I}(\bar{\varepsilon})$  as the inverse image of the tip,  $\tau_F$  under  $F$ , that is,  $c_F = F^{-1}(\tau_F)$ . Let the tip and the critical point of  $R^k F$  be  $\tau_k$  and  $c_{F_k}$  respectively. Recall the definition of tip

$$\{\tau_k\} = \bigcap_{n \geq k+1} \Psi_{k, \mathbf{v}}^n(B)$$

where  $B$  is the domain,  $B(R^n F)$  for each  $n \in \mathbb{N}$ . The above intersection is nested and each  $\Psi_{k, \mathbf{v}}^n(B)$  is connected. Then the tip is just the limit of the sequence of  $\Psi_{k, \mathbf{v}}^n(B)$  as follows

$$(4.1) \quad \{\tau_k\} = \bigcap_{n \geq 1} \Psi_{k, \mathbf{v}}^n(B) = \lim_{n \rightarrow \infty} \Psi_{k, \mathbf{v}}^n(B).$$

**Lemma 4.1.** *Let  $F$  be the Hénon-like map in  $\mathcal{I}(\bar{\varepsilon})$ . Then the critical point of  $F$ ,  $c_F$  is the following limit*

$$\{c_{F_k}\} = \lim_{n \rightarrow \infty} \Psi_{k, \mathbf{c}}^n(B)$$

where  $B$  is the domain,  $B(R^n F)$  for  $k < n$ .

*Proof.* Since  $\text{diam}(\Psi_{\mathbf{w}}^n) \leq C\sigma^{n-k}$  for some  $C > 0$ , the limit of  $\Psi_{\mathbf{w}_n}^n(B)$  as  $n \rightarrow \infty$  is a single point. By Corollary 3.4, the following equation holds

$$\Psi_{k, \mathbf{v}}^n \circ F_n = F_k \circ \Psi_{k, \mathbf{c}}^n$$

for each  $k < n$ . Observe that  $\psi_v^{n+1}(B(R^{n+1}F)) \subset B(R^n F)$  and  $\psi_c^{n+1}(B(R^{n+1}F)) \subset B(R^n F)$ . Then passing the limit, the following equation holds

$$(4.2) \quad \begin{aligned} F_k \circ \lim_{n \rightarrow \infty} \Psi_{k, \mathbf{c}}^n(B(R^n F)) &= \lim_{n \rightarrow \infty} F_k \circ \Psi_{k, \mathbf{c}}^n(B(R^n F)) \\ &= \lim_{n \rightarrow \infty} F_k \circ \Psi_{k, \mathbf{c}}^n \circ \psi_v^{n+1}(B(R^{n+1}F)) \\ &= \lim_{n \rightarrow \infty} \Psi_{k, \mathbf{v}}^n \circ \psi_c^{n+1}(B(R^{n+1}F)) \\ &= \lim_{n \rightarrow \infty} \Psi_{k, \mathbf{v}}^n(B(R^n F)) = \{\tau_k\} \end{aligned}$$

for all  $n \in \mathbb{N}$  because each limit is a single point set. Then the critical point of  $F$ ,  $\{c_{F_k}\}$  is  $\lim_{n \rightarrow \infty} \Psi_{k, \mathbf{c}}^n(B)$ .  $\square$

**Proposition 4.2.** *Let  $F$  be the Hénon-like map in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Then the following equation holds*

$$\partial_x \delta_n(w) = \partial_x \delta_k \circ \Psi_{k, \mathbf{c}}^n(w) - \sum_{i=k}^{n-1} q_i \circ (\pi_x \circ \Psi_{i, \mathbf{c}}^n(w))$$

where  $\delta_n(w)$  is the third coordinate map of  $R^n F$  for each  $n \in \mathbb{N}$ . Moreover, passing the limit the following equation holds

$$\partial_x \delta(c_{F_k}) = \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_x \circ \Psi_{i, \mathbf{c}}^n(w)) = \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i(\pi_x(c_{F_i}))$$

where  $c_{F_k}$  is the critical point of  $R^k F$  for  $k < n$ .



*Proof.* By Lemma 3.2, we see

$$\partial_x \delta_n(w) = \partial_x \delta_{n-1} \circ \psi_c^n(w) - q_{n-1} \circ (\sigma_{n-1} \cdot x)$$

Recall the definition of  $q_k(x)$  in the equation (3.1). Then inductively we obtain that

$$\begin{aligned} \partial_x \delta_n(w) &= \partial_x \delta_{n-1} \circ \psi_c^n(w) - q_{n-1}(\pi_x \circ \psi_c^n(w)) \\ &= \partial_x \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_c^n(w)) - q_{n-2} \circ (\pi_x \circ \psi_c^{n-1} \circ \psi_c^n(w)) - q_{n-1} \circ (\pi_x \circ \psi_c^n(w)) \\ &\quad \vdots \\ &= \partial_x \delta_k \circ \Psi_{k,\mathbf{c}}^n(w) - \sum_{i=k}^{n-1} q_i \circ (\pi_x \circ \Psi_{i,\mathbf{c}}^n(w)). \end{aligned}$$

Recall that  $\|\partial_x \delta_n\| \leq C\varepsilon^{2^n}$  for some  $C > 0$  and  $\lim_{n \rightarrow \infty} \Psi_{i,\mathbf{c}}^n(B) = \{c_{F_i}\}$  for each  $i < n$ . Thus passing the limit, we obtain

$$\partial_x \delta_k(c_F) = \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_x \circ \Psi_{i,\mathbf{c}}^n(w)).$$

for  $w \in \text{Dom}(R^n F)$ . Since  $\partial_x \delta_k(c_F)$  is the single point for each  $k$  and the critical points of each level,  $c_{F_i}$  are contained in  $\Psi_{i,\mathbf{c}}^n(B)$  for all  $n \in \mathbb{N}$ . Then

$$\partial_x \delta_k(c_{F_k}) = \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i(\pi_x(c_{F_i})).$$

□

#### 4.2. Universal number $b_2$ and the asymptotic of $\partial_z \delta_n$ and $\partial_y \delta_n$ .

**Proposition 4.3.** *Let  $F$  be Hénon-like diffeomorphism in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Let  $\delta_n$  be the third coordinate map of  $R^n F$  for  $n \in \mathbb{N}$ . Then*

$$\partial_z \delta_n = b_2^{2^n} (1 + O(\rho^n))$$

for each  $n \in \mathbb{N}$  and  $0 < \rho < 1$  where  $b_2$  is a non zero number.

*Proof.* The recursive formula of  $\partial_z \delta_n$  in Lemma 3.2 implies the following equation by inductive calculation

$$\begin{aligned} \partial_z \delta_n(w) &= \partial_z \delta_{n-1} \circ (F_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1} w)) \cdot \partial_z \delta_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1} w) \\ &= \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_z \delta_{n-1} \circ \psi_v^n(w) \\ &= \partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_c^n(w)) \cdot \partial_z \delta_{n-2} \circ (\psi_v^{n-1} \circ \psi_c^n(w)) \\ &\quad \cdot \partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_v^n(w)) \cdot \partial_z \delta_{n-2} \circ (\psi_v^{n-1} \circ \psi_v^n(w)) \\ &\quad \vdots \\ &= \prod_{\mathbf{w} \in W^n} \partial_z \delta \circ \Psi_{\mathbf{w}}^n(w). \end{aligned} \tag{4.3}$$

where  $\mathbf{w}$  is the word of length  $n$  in  $W^n = \{v\ c\}^n$ . The number of words in  $W^n$  is  $2^n$ . Let us take the logarithmic average of  $|\partial_z \delta_n|$  on the regions  $\Psi_{\mathbf{w}}^n(B)$  and let this map be  $l_n(w)$  for each  $n \in \mathbb{N}$ <sup>2</sup>

$$(4.4) \quad l_n(w) = \frac{1}{2^n} \sum_{\mathbf{w} \in W^n} \log |\partial_z \delta \circ \Psi_{\mathbf{w}}^n(w)|.$$

The limit of  $l_n(w)$  as  $n \rightarrow \infty$  is a function defined on the critical Cantor set,  $\mathcal{O}_F$  as  $n \rightarrow \infty$ . However, values of the limit function at all points of  $\mathcal{O}_F$  are the same as each other, that is, the limit is a constant function. In particular, we have

$$l_n(w) \rightarrow \int_{\mathcal{O}_F} \log |\partial_z \delta| d\mu.$$

where  $\mu$  is the unique ergodic probability measure on  $\mathcal{O}_F$ . Let this limit be  $\log b_2$  for some  $b_2 > 0$ . Since  $\text{diam}(\Psi_{\mathbf{w}}^n(B)) \leq C\sigma^n$  for some  $C > 0$  and for all  $\mathbf{w} \in W^n$ , the above equation (4.4) converges exponentially fast as  $n \rightarrow \infty$ . In other words,

$$\frac{1}{2^n} \log |\partial_z \delta_n(w)| = \log b_2 + O(\rho_0^n)$$

for some  $0 < \rho_0 < 1$ . Let us choose the constant  $\rho = \rho_0/2$ . Then we obtain the following equation

$$(4.5) \quad \begin{aligned} \log |\partial_z \delta_n(w)| &= 2^n \log b_2 + O(\rho^n) \\ &= 2^n \log b_2 + \log(1 + O(\rho^n)) \\ &= \log b_2^{2^n} (1 + O(\rho^n)). \end{aligned}$$

Hence,

$$(4.6) \quad |\partial_z \delta_n| = b_2^{2^n} (1 + O(\rho^n)).$$

We may assume that  $\partial_z \delta$  is non zero. The proof is complete.  $\square$

**Lemma 4.4.** *Let  $F$  be the Hénon-like diffeomorphism in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Let  $\delta_n(w)$  be the third coordinate map of  $F_n \equiv R^n F$  for each  $n \in \mathbb{N}$ . Then the following equation holds*

$$\begin{aligned} &\partial_y \delta_n(w) \cdot \partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) \\ &= \partial_z \delta_n(w) \cdot \left[ \partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) + \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \cdot \partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) \right] \end{aligned}$$

for  $k < n$ . Moreover,

$$\partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) \cdot [\partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w)]^{-1} + \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \leq C\sigma^{n-k}$$

for some  $C > 0$  and  $0 < \rho < 1$ .

---

<sup>2</sup>If  $\partial_z \delta(w) = 0$  for some  $w \in B$ , then  $\partial_y \delta(w) = 0$  at the same point because  $F \in \mathcal{N}$ . Thus  $\text{Jac } F(w) = 0$ , that is,  $F$  cannot be a diffeomorphism. Moreover,  $\partial_z \delta$  is defined on some compact set which contains the set  $\bigcup_{\mathbf{w} \in W^n} \Psi_{\mathbf{w}}^n(B)$ . Then we may assume that  $\partial_z \delta(w)$  has the positive lower bounds (or negative upper bounds) on  $B_v^1 \cup B_c^1$ .

*Proof.* Recall the recursive formula in Lemma 3.2

$$\begin{aligned}\partial_y \delta_n(w) &= \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_y \delta_{n-1} \circ \psi_v^n(w) + \partial_z \delta_n(w) \cdot q_{n-1} \circ (\pi_y \circ \psi_v^n(w)) \\ \partial_z \delta_n(w) &= \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_z \delta_{n-1} \circ \psi_v^n(w).\end{aligned}$$

Then by the inductive calculation, we have the following equation

$$\begin{aligned}\partial_y \delta_n(w) &= \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_y \delta_{n-1} \circ \psi_v^n(w) + \partial_z \delta_n(w) \cdot q_{n-1} \circ (\pi_y \circ \psi_v^n(w)) \\ &= \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \left[ \partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_v^n(w)) \cdot \partial_y \delta_{n-2} \circ (\psi_v^{n-1} \circ \psi_v^n(w)) \right. \\ &\quad \left. + \partial_z \delta_{n-1} \circ \psi_v^n(w) \cdot q_{n-2} \circ (\pi_y \circ (\psi_v^{n-1} \circ \psi_v^n(w))) \right] + \partial_z \delta_n(w) \cdot q_{n-1} \circ (\pi_y \circ \psi_v^n(w)) \\ &= \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_v^n(w)) \cdot \partial_y \delta_{n-2} \circ (\psi_v^{n-1} \circ \psi_v^n(w)) \\ &\quad + \partial_z \delta_n(w) \cdot \left[ q_{n-2} \circ (\pi_y \circ (\psi_v^{n-1} \circ \psi_v^n(w))) + q_{n-1} \circ (\pi_y \circ \psi_v^n(w)) \right] \\ &\quad \vdots \\ &= \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_v^n(w)) \cdots \partial_z \delta_k \circ (\psi_c^{k+1} \circ \psi_v^{k+2} \circ \cdots \circ \psi_v^n(w)) \\ &\quad \cdot \partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) + \partial_z \delta_n(w) \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)).\end{aligned}$$

Thus let us multiply  $\partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w)$  to both sides. Then we obtain that

$$\begin{aligned}\partial_y \delta_n(w) \cdot \partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) &= \partial_z \delta_n(w) \cdot \partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) + \partial_z \delta_n(w) \cdot \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \cdot \partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) \\ &= \partial_z \delta_n(w) \cdot \left[ \partial_y \delta \circ \Psi_{k,\mathbf{v}}^n(w) + \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \cdot \partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) \right]\end{aligned}$$

for  $k < n$ . Since  $F_k$  is a diffeomorphism, there exists  $[\partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w)]^{-1}$  for all  $w \in B(R^n F)$ . Then

$$\partial_y \delta_n(w) = \partial_z \delta_n(w) \cdot \left[ (\partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w))^{-1} \cdot \partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) + \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \right]$$

Let us estimate the second factor the right side of the above equation

$$(4.7) \quad (\partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w))^{-1} \cdot \partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) + \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)).$$

Since  $F_i(c_{F_i}) = \tau_i$  and Hénon-like map  $F_i$  is  $(f_i(x) - \varepsilon_i(w), x, \delta_i(w))$ , observe that

$$\pi_x(c_{F_i}) = \pi_y(\tau_i)$$

for every  $i \in \mathbb{N}$ . By Proposition 4.2, we have the following equation

$$\partial_x \delta_k(c_F) = \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_x \circ \Psi_{i,\mathbf{c}}^n(w))$$

and it converges exponentially fast. Recall the fact that

$$(4.8) \quad \pi_x \circ \Psi_{i,\mathbf{c}}^n(w) = \sigma_{n,i} x$$

$$(4.9) \quad \pi_y \circ \Psi_{i,\mathbf{c}}^n(w) = \sigma_{n,i} y$$

for  $i < n$ . Then the expression (4.9) converges with the same rate of the expression (4.8). Take the limit of (4.7).

$$\begin{aligned} & \lim_{n \rightarrow \infty} (\partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w))^{-1} \cdot \partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) + \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \\ &= (\partial_z \delta_k(\tau_k))^{-1} \cdot \partial_y \delta_k(\tau_k) + \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_y(\tau_i)) \\ &= (\partial_z \delta_k(\tau_k))^{-1} \cdot \partial_y \delta_k(\tau_k) + \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_x(c_{F_i})) \\ &= (\partial_z \delta_k(\tau_k))^{-1} \cdot \partial_y \delta_k(\tau_k) + \partial_x \delta_k(c_{F_k}) \\ &= 0 \end{aligned}$$

Recall that  $\text{diam } \Psi_{k,\mathbf{w}}^n(B) \leq C\sigma^{n-k}$  for some  $c > 0$ . Thus  $\Psi_{k,\mathbf{v}}^n(B)$  and  $\Psi_{k,\mathbf{c}}^n(B)$  converge to  $\tau_F$  and  $c_F$  respectively as  $n \rightarrow \infty$  at the same rate. Then

$$(4.10) \quad (\partial_z \delta_k \circ \Psi_{k,\mathbf{v}}^n(w))^{-1} \cdot \partial_y \delta_k \circ \Psi_{k,\mathbf{v}}^n(w) + \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \leq C\sigma^{n-k}$$

for some  $C > 0$ . □

**4.3. Universal number  $b_1$  and  $\partial_y \varepsilon$ .** In Section 4.2, the universal number  $b_2$  represents asymptotic for  $\partial_z \delta$ . Universality of Jacobian determinant implies that average Jacobian of  $F$ ,  $b_F$  is the universal number. Define the number  $b_1$  as the ratio  $b_F/b_2$  and we would show that  $b_1$  is also the universal number which describes the asymptotic of  $\partial_y \varepsilon$ .

Suppose that Hénon-like diffeomorphism  $F$  is in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . By the universal expression of Jacobian determinant with average Jacobian and Proposition 4.3, we obtain that

$$\begin{aligned} \text{Jac } F_k(w) &= b_F^{2^k} a(x)(1 + O(\rho^k)) \\ &= \partial_y \varepsilon_k(w) \cdot \partial_z \delta_k(w) - \partial_z \varepsilon_k(w) \cdot \partial_y \delta_k(w) \\ &= [\partial_y \varepsilon_k(w) - \partial_z \varepsilon_k(w) \cdot \partial_y \delta_k(w) \cdot (\partial_z \delta_k(w))^{-1}] \cdot \partial_z \delta_k(w) \\ &= [\partial_y \varepsilon_k(w) - \partial_z \varepsilon_k(w) \cdot \partial_y \delta_k(w) \cdot (\partial_z \delta_k(w))^{-1}] \cdot b_2^{2^k} (1 + O(\rho^k)). \end{aligned}$$

The above equation implies the existence of another universal number  $b_F/b_2$ .

**Lemma 4.5.** *Let  $F$  be the Hénon-like map in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Then there exists the number  $b_1 \equiv b_F/b_2$  satisfying the following equation*

$$\partial_y \varepsilon_k(w) - \partial_z \varepsilon_k(w) \cdot \partial_y \delta_k(w) \cdot (\partial_z \delta_k(w))^{-1} = b_1^{2^k} a(x)(1 + O(\rho^k))$$

for each  $k \in \mathbb{N}$ .

**Lemma 4.6.** *Let  $F$  be the Hénon-like diffeomorphism in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Then the following equation holds for  $k < n$*

$$\partial_y \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) + \partial_z \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \cdot \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \leq C_1 b_1^{2^k} + C_2 \bar{\varepsilon}^{2^k} \sigma^{n-k}$$

where  $w \in B(R^n F)$  for some positive  $C_1$  and  $C_2$ .

*Proof.* The equation (4.2) and (4.5) implies that

$$\begin{aligned} & b_1^{2^k} a \circ (\Psi_{k,\mathbf{v}}^n(w))(1 + O(\rho^k)) \\ &= \partial_y \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) - \partial_z \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \cdot \partial_y \delta_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \cdot (\partial_z \delta_k \circ (\Psi_{k,\mathbf{v}}^n(w)))^{-1} \\ &= \partial_y \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \\ & \quad - \partial_z \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \cdot \left[ - \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) + (\partial_z \delta_n(w))^{-1} \cdot \partial_y \delta_n(w) \right] \\ &= \partial_y \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) + \partial_z \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \cdot \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \\ & \quad - \partial_z \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \cdot (\partial_z \delta_n(w))^{-1} \cdot \partial_y \delta_n(w) \end{aligned}$$

Lemma 4.4 implies that

$$\|\partial_z \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w))\| \cdot \|(\partial_z \delta_n(w))^{-1} \cdot \partial_y \delta_n(w)\| \leq C_2 \bar{\varepsilon}^{2^k} \sigma^{n-k}$$

for some  $C_2 > 0$  independent of  $k$ . Hence,

$$\partial_y \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) + \partial_z \varepsilon_k \circ (\Psi_{k,\mathbf{v}}^n(w)) \cdot \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) \leq C_1 b_1^{2^k} + C_2 \bar{\varepsilon}^{2^k} \sigma^{n-k}$$

□

## 5. Recursive formula of $\Psi_k^n$

In this section, let us calculate recursive formulas of some components of  $D\Psi_k^n$ . Recall that  $\Psi_k^n$  is the conjugation between  $F_k^{2^{n-k}}$  and  $F_n$ . These formulas in this section would be used in the estimation of minimal distances of a particular adjacent boxes and the diameter of boxes in the next sections.

**Lemma 5.1.** *Let  $F \in \mathcal{I}(\bar{\varepsilon})$ . The derivative of non-linear conjugation  $\Psi_k^n$  at the tip,  $\tau_{F_k}$  between  $F_k^{2^{n-k}}$  and  $F_n$  is as follows*

$$D_k^n \equiv D\Psi_k^n(\tau_n) = \begin{pmatrix} \alpha_{n,k} & \sigma_{n,k} t_{n,k} & \sigma_{n,k} u_{n,k} \\ & \sigma_{n,k} & \\ & \sigma_{n,k} d_{n,k} & \sigma_{n,k} \end{pmatrix}$$

where  $\sigma_{n,k}$  and  $\alpha_{n,k}$  are linear scaling factors such that  $\sigma_{n,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$  and  $\alpha_{n,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$ . Then

$$\begin{aligned} d_{n,k} &= \sum_{i=k}^{n-1} d_{i+1,i}, \quad u_{n,k} = \sum_{i=k}^{n-1} \sigma^{i-k} u_{i+1,i} (1 + O(\rho^k)) \\ t_{n,k} &= \sum_{i=k}^{n-1} \sigma^{i-k} [t_{i+1,i} + u_{i+1,i} d_{n,i+1}] (1 + O(\rho^k)) \\ t_{n,k} - u_{n,k} d_{n,k} &= \sum_{i=k}^{n-1} \sigma^{i-k} [t_{i+1,i} - u_{i+1,i} d_{i+1,k}] (1 + O(\rho^k)) \end{aligned}$$

where  $\sigma^{i-k}(1 + O(\rho^k)) = \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}}$ . Moreover,  $d_{n,k}$ ,  $u_{n,k}$  and  $t_{n,k}$  are convergent as  $n \rightarrow \infty$  super exponentially fast.

*Proof.*  $D_k^n = D_k^m \cdot D_m^n$  for any  $m$  between  $k$  and  $n$  because  $\Psi_k^n(\tau_{F_n})$  is  $\tau_{F_k}$ , the tip of  $k^{th}$  level. By the direct calculation, we obtain that

$$\begin{aligned} & D_k^m \cdot D_m^n \\ &= \begin{pmatrix} \alpha_{n,k} & \boxed{\alpha_{m,k} \sigma_{n,m} t_{n,m} + \sigma_{n,k} t_{m,k} + \sigma_{n,k} u_{m,k} d_{n,m}} & \boxed{\alpha_{m,k} \sigma_{n,m} u_{n,m} + \sigma_{n,k} u_{m,k}} \\ & \sigma_{n,k} & \\ & \boxed{\sigma_{n,k} d_{m,k} + \sigma_{n,k} d_{n,m}} & \sigma_{n,k} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \sigma_{n,k} t_{n,k} &= \alpha_{m,k} \sigma_{n,m} t_{n,m} + \sigma_{n,k} t_{m,k} + \sigma_{n,k} u_{m,k} d_{n,m} \\ \sigma_{n,k} u_{n,k} &= \alpha_{m,k} \sigma_{n,m} u_{n,m} + \sigma_{n,k} u_{m,k} \\ \sigma_{n,k} d_{n,k} &= \sigma_{n,k} d_{m,k} + \sigma_{n,k} d_{n,m} \end{aligned}$$

for any  $m$  between  $k$  and  $n$ . Recall that  $\sigma_{n,k} = \sigma_{n,m} \cdot \sigma_{m,k}$  and  $\alpha_{n,k} = \alpha_{n,m} \cdot \alpha_{m,k}$ . Let  $m$  be  $k+1$ . Then

$$\begin{aligned} d_{n,k} &= d_{n,k+1} + d_{k+1,k} \\ &= d_{n,k+2} + d_{k+2,k+1} + d_{k+1,k} \\ &\quad \vdots \\ &= d_{n,n-1} + \cdots + d_{k+2,k+1} + d_{k+1,k} \end{aligned}$$

$$= \sum_{i=k}^{n-1} d_{i+1,i}.$$

Each term is bounded by  $\varepsilon^{2^i}$  for each  $i$ , that is,  $|d_{i+1,i}| \asymp |q_i(\pi_y(\tau_{i+1}))| \leq \|D\delta_i\| = O(\varepsilon^{2^i})$ . Then  $d_{n,k}$  converges to the number, say  $d_{*,k}$  super exponentially fast.

Let us see the recursive formula of  $u_{n,k}$

$$\begin{aligned} u_{n,k} &= \frac{\alpha_{k+1,k}}{\sigma_{k+1,k}} u_{n,k+1} + u_{k+1,k} \\ &= \frac{\alpha_{k+1,k}}{\sigma_{k+1,k}} \left[ \frac{\alpha_{k+2,k+1}}{\sigma_{k+2,k+1}} u_{n,k+2} + u_{k+2,k+1} \right] + u_{k+1,k} \\ &\quad \vdots \\ &= \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}} u_{i+1,i} + u_{k+1,k} \\ &= \sum_{i=k}^{n-1} \sigma^{i-k} u_{i+1,i} (1 + O(\rho^k)). \end{aligned}$$

Since  $u_{i+1,i} \asymp \partial_z \varepsilon_i(\tau_{F_{i+1}})$ ,  $u_{n,k}$  converges to the number, say  $u_{*,k}$  also super exponentially fast. Let us see the recursive formula of  $t_{n,k}$

$$\begin{aligned} t_{n,k} &= \frac{\alpha_{k+1,k}}{\sigma_{k+1,k}} t_{n,k+1} + t_{k+1,k} + u_{k+1,k} d_{n,k+1} \\ &= \frac{\alpha_{k+1,k}}{\sigma_{k+1,k}} \left[ \frac{\alpha_{k+2,k+1}}{\sigma_{k+2,k+1}} t_{n,k+2} + t_{k+2,k+1} + u_{k+2,k+1} d_{n,k+2} \right] + t_{k+1,k} + u_{k+1,k} d_{n,k+1} \\ &\quad \vdots \\ &= \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}} t_{i+1,i} + t_{k+1,k} + \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}} u_{i+1,i} d_{n,i+1} + u_{k+1,k} d_{n,k+1} \\ &= \sum_{i=k}^{n-1} \sigma^{i-k} [t_{i+1,i} + u_{i+1,i} d_{n,i+1}] (1 + O(\rho^k)). \end{aligned}$$

By the above equations for  $d_{n,k}$ ,  $u_{n,k}$  and  $t_{n,k}$ , we obtain the recursive formula of  $t_{n,k} - u_{n,k} d_{n,k}$  as follows

$$\begin{aligned} &t_{n,k} - u_{n,k} d_{n,k} \\ &= \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}} [t_{i+1,i} + u_{i+1,i} d_{n,i+1}] + t_{k+1,k} + u_{k+1,k} d_{n,k+1} \\ &\quad - \left[ \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}} u_{i+1,i} + u_{k+1,k} \right] d_{n,k} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}} [t_{i+1,i} + u_{i+1,i} d_{n,i+1} - u_{i+1,i} d_{n,k}] + t_{k+1,k} + u_{k+1,k} d_{n,k+1} - u_{k+1,k} d_{n,k} \\
&= \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,j}}{\sigma_{j+1,j}} [t_{i+1,i} - u_{i+1,i} d_{i+1,k}] + t_{k+1,k} - u_{k+1,k} d_{k+1,k} \\
&= \sum_{i=k}^{n-1} \sigma^{i-k} [t_{i+1,i} - u_{i+1,i} d_{i+1,k}] (1 + O(\rho^k)).
\end{aligned}$$

Recall the derivative of coordinate change map at the tip on each level

$$\sigma_k \cdot DH_k(\tau_{F_k}) = (D_k^{k+1})^{-1} = \begin{pmatrix} (\alpha_k)^{-1} & & \\ & (\sigma_k)^{-1} & \\ & & (\sigma_k)^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & -t_k + u_k d_k & -u_k \\ & 1 & \\ & -d_k & 1 \end{pmatrix}.$$

Since  $H_k(w) = (f_k(x) - \varepsilon_k(w), y, z - \delta_k(y, f_k^{-1}(y), 0))$ , we see that  $\partial_y \varepsilon_k(\tau_{F_k}) \asymp -t_k + u_k d_k$  for every  $k \in \mathbb{N}$ . Moreover, the fact that  $t_{i+1,i} - u_{i+1,i} d_{i+1,k} \asymp \partial_y \varepsilon_i(\tau_{F_{i+1}})$  and  $|u_{i+1,i} d_{n,i}|$  is super exponentially small for each  $i < n$  implies that  $t_{n,k}$  converges to a number, say  $t_{*,k}$  super exponentially fast.  $\square$

Recall the expression of  $\Psi_k^n$  from  $B(R^n F)$  to  $B_{\mathbf{v}}^{n-k}(R^k F)$

$$\Psi_k^n(w) = \begin{pmatrix} 1 & t_{n,k} & u_{n,k} \\ & 1 & \\ & d_{n,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,k} & & \\ & \sigma_{n,k} & \\ & & \sigma_{n,k} \end{pmatrix} \begin{pmatrix} x + S_k^n(w) \\ y \\ z + R_k^n(y) \end{pmatrix}$$

where  $\mathbf{v} = v^{n-k} \in W^{n-k}$ . Recall that  $\Psi_k^n$  be the map from  $B(R^n F)$  to  $B(R^k F)$  as the conjugation between  $(R^k F)^{2^{n-k}}$  and  $R^n F$ .

**Lemma 5.2.** *Let  $F \in \mathcal{I}(\bar{\varepsilon})$ . Then both  $R_k^n(y)$  and  $(R_k^n)'(y)$  converges to zero exponentially fast as  $n \rightarrow \infty$  where  $R_k^n(y)$  be non-linear part of  $\pi_z \circ \Psi_k^n$  depending only on the second variable  $y$ .*

*Proof.* Let  $w = (x, y, z)$  be the point in  $B(R^n F)$  and let  $\Psi_{n-1}^n(w)$  be  $w' = (x', y', z')$ . Recall  $\Psi_k^n = \Psi_{n-1}^n \circ \Psi_k^{n-1}$ . Thus

$$\begin{aligned}
z' &= \pi_z \circ \Psi_{n-1}^n(w) = \sigma_{n,n-1} [d_{n,n-1} y + z + R_{n-1}^n(y)] \\
y' &= \pi_y \circ \Psi_{n-1}^n(w) = \sigma_{n,n-1} y.
\end{aligned}$$

Then by the composition of  $\Psi_k^{n-1}$  and  $\Psi_{n-1}^n$ , we obtain the recursive formula of  $\pi_z \circ \Psi_k^n$  as follows

$$\begin{aligned}
(5.1) \quad &\pi_z \circ \Psi_k^n(w) \\
&= \sigma_{n,k} [d_{n,k} y + z + R_k^n(y)] \\
&= \pi_z \circ \Psi_k^{n-1}(w') = \sigma_{n-1,k} [d_{n-1,k} y' + z' + R_k^{n-1}(y')]
\end{aligned}$$



$$\begin{aligned}
&= \sigma_{n-1,k} \left[ d_{n-1,k} \sigma_{n,n-1} y + \sigma_{n,n-1} \left[ d_{n,n-1} y + z + R_{n-1}^n(y) \right] + R_k^{n-1}(\sigma_{n,n-1} y) \right] \\
(5.2) \quad &= \sigma_{n,k} (d_{n-1,k} + d_{n,n-1}) + \sigma_{n,k} z + \sigma_{n,k} R_{n-1}^n(y) + \sigma_{n-1,k} R_k^{n-1}(\sigma_{n,n-1} y).
\end{aligned}$$

By Proposition 5.1,  $d_{n,k} = d_{n-1,k} + d_{n,n-1}$ . Let us compare (5.1) with (5.2). Recall the equation  $\sigma_{n,k} = \sigma_{n,n-1} \cdot \sigma_{n-1,k}$ . Then

$$R_k^n(y) = R_{n-1}^n(y) + \frac{1}{\sigma_{n,n-1}} R_k^{n-1}(\sigma_{n,n-1} y).$$

Each  $R_j^i(y)$  is the sum of the second and higher order terms of  $\pi_z \circ \Psi_j^i$  for  $i > j$ . Thus

$$R_k^n(y) = a_{n,k} y^2 + A_{n,k}(y) \cdot y^3$$

Moreover,  $\|R_{n-1}^n\| = O(\bar{\varepsilon}^{2^{n-1}})$  because  $R_{n-1}^n(y)$  is the second and higher order terms of the map  $\delta_{n-1}(\sigma_{n,n-1} y, f_{n-1}^{-1}(\sigma_{n,n-1} y), 0)$ . Then

$$R_k^n(y) = \frac{1}{\sigma_{n,n-1}} R_k^{n-1}(\sigma_{n,n-1} y) + c_{n,k} y^2 + O(\bar{\varepsilon}^{2^{n-1}} y^3)$$

where  $c_{n,k} = O(\bar{\varepsilon}^{2^{n-1}})$ . The recursive formula for  $a_{n,k}$  and  $A_{n,k}$  as follows

$$R_k^n(y) = \frac{1}{\sigma_{n,n-1}} \left( a_{n-1,k} \cdot (\sigma_{n,n-1} y)^2 + A_{n-1,k}(\sigma_{n,n-1} y) \cdot (\sigma_{n,n-1} y)^3 \right) + O(\bar{\varepsilon}^{2^{n-1}} y^3).$$

Then  $a_{n,k} = \sigma_{n,n-1} a_{n-1,k} + c_{n,k}$  and  $\|A_{n,k}\| \leq \|\sigma_{n,n-1}\|^2 \|A_{n-1,k}\| + O(\bar{\varepsilon}^{2^{n-1}})$  and for each fixed  $k < n$ ,  $a_{n,k} \rightarrow 0$  and  $A_{n,k} \rightarrow 0$  exponentially fast as  $n \rightarrow \infty$ . Thus  $R_k^n(y)$  converges to zero as  $n \rightarrow \infty$  exponentially fast. Let us estimate  $\|A'_{n,k}\|$  in order to measure how fast  $(R_k^n)'(y)$  is convergent. By similar method, we have the recursive formula of  $(R_k^n)'(y)$  as follows

$$(R_k^n)'(y) = 2 a_{n,k} y + 3 A_{n,k}(y) \cdot y^2 + A'_{n,k}(y) \cdot y^3$$

$$\begin{aligned}
\text{Thus } (R_k^n)'(y) &= (R_{n-1}^n)'(y) + R_k^{n-1}(\sigma_{n,n-1} y) \\
&= R_k^{n-1}(\sigma_{n,n-1} y) + 2 c_{n,k} y + O(\bar{\varepsilon}^{2^{n-1}} y^2).
\end{aligned}$$

Then

$$\begin{aligned}
(R_k^n)'(y) &= 2 a_{n-1,k} \sigma_{n,n-1} y + 3 A_{n-1,k}(\sigma_{n,n-1} y) \cdot (\sigma_{n,n-1} y)^2 + A'_{n,k}(\sigma_{n,n-1} y) \cdot (\sigma_{n,n-1} y)^3 \\
&\quad + 2 c_{n,k} y + O(\bar{\varepsilon}^{2^{n-1}} y^2).
\end{aligned}$$

Let us compare quadratic and higher order terms of  $(R_k^n)'(y)$

$$\begin{aligned}
3 A_{n,k}(y) \cdot y^2 + A'_{n,k}(y) \cdot y^3 &= 3 A_{n-1,k}(\sigma_{n,n-1} y) \cdot (\sigma_{n,n-1} y)^2 + A'_{n,k}(\sigma_{n,n-1} y) \cdot (\sigma_{n,n-1} y)^3 \\
&\quad + O(\bar{\varepsilon}^{2^{n-1}} y^2).
\end{aligned}$$

Thus

$$A'_{n,k}(y) y = A'_{n,k}(\sigma_{n,n-1} y) \cdot \sigma_{n,n-1}^3 y - 3 A_{n,k}(y) + 3 A_{n-1,k}(\sigma_{n,n-1} y) \cdot \sigma_{n,n-1}^2 + O(\bar{\varepsilon}^{2^{n-1}}).$$

Then

$$\begin{aligned}
\|A'_{n,k}\| &\leq \|A'_{n-1,k}\| \cdot \|\sigma_{n,n-1}\|^3 + 3\|A_{n,k}\| + 3\|A_{n-1,k}\| \cdot \|\sigma_{n,n-1}\|^2 + O(\bar{\varepsilon}^{2^{n-1}}) \\
&\leq \|A'_{n-1,k}\| \cdot \|\sigma_{n,n-1}\|^3 + C\|\sigma_{n,n-1}\|^2
\end{aligned}$$

for some  $C > 0$ . Then  $A'_{n,k} \rightarrow 0$  as  $n \rightarrow \infty$  exponentially fast. Hence, so does  $(R_k^n)'(y)$  exponentially fast.  $\square$

Let  $w^1$  and  $w^2$  be two points in  $B(R^n F)$  and  $w^j = (x^j, y^j, z^j)$  for  $j = 1, 2$ . Let  $\Psi_{i,\mathbf{v}}^n(w^j) = w_i^j$  for  $i \in \mathbb{N}$  and  $j = 1, 2$ .

**Proposition 5.3.** *Let  $F \in \mathcal{I}(\bar{\varepsilon})$ . Then*

$$\dot{z}^1 - \dot{z}^2 = \pi_z \circ \Psi_k^n(w^1) - \pi_z \circ \Psi_k^n(w^2) = \sigma_{n,k} \cdot (z^1 - z^2) + \sigma_{n,k} \sum_{i=k}^{n-1} q_i(\sigma_{n,i} \bar{y}) \cdot (y^1 - y^2)$$

where  $\bar{y}$  is in the line segment between  $y_1$  and  $y_2$ . Moreover,

$$\sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \bar{y}) \cdot (y^1 - y^2) = d_{n,k} \cdot (y^1 - y^2) + R_k^n(y^1) - R_k^n(y^2).$$

*Proof.* Firstly, let us express  $\pi_z \circ \Psi_k^n(w)$ . Let  $p_i(y)$  be  $\delta_i(y, f_i^{-1}(y), 0)$  in order to simplify expression. Let  $\Psi_{i,\mathbf{v}}^n(w) = w_i$  for  $k \leq i \leq n-1$  and let  $w_i = (x_i, y_i, z_i)$ <sup>3</sup>. Let  $w = w_n$ . Recall  $\pi_z \circ \psi_i^{i+1}(w_{i+1}) = \sigma_i z_{i+1} + p_i(\sigma_i y_{i+1})$ . Since  $\Psi_k^n = \psi_k^{k+1} \circ \Psi_{k+1}^n$ , we estimate  $z_k$  using recursive formula

$$\begin{aligned} z_k &= \pi_z \circ \Psi_k^n(w) = \pi_z \circ \psi_k^{k+1}(w_{k+1}) \\ &= \sigma_k \cdot z_{k+1} + p_k(\sigma_k \cdot y_{k+1}) \\ &= \sigma_k [\sigma_{k+1} \cdot z_{k+2} + p_{k+1}(\sigma_{k+1} \cdot y_{k+2})] + p_k(\sigma_k \cdot y_{k+1}) \\ &= \sigma_k \sigma_{k+1} \cdot z_{k+2} + \sigma_k \cdot p_{k+1}(\sigma_{k+1} \cdot y_{k+2}) + p_k(\sigma_k \cdot y_{k+1}) \\ &\quad \vdots \\ &= \sigma_k \sigma_{k+1} \cdots \sigma_{n-1} \cdot z + [\sigma_k \sigma_{k+1} \cdots \sigma_{n-2} \cdot p_{n-1}(\sigma_{n-1} \cdot y) \\ &\quad + \sigma_k \sigma_{k+1} \cdots \sigma_{n-3} \cdot p_{n-2}(\sigma_{n-2} \cdot y_{n-1}) + \cdots + p_k(\sigma_k \cdot y_{k+1})] \\ &= \sigma_{n,k} \cdot z + \sigma_{n-1,k} \cdot p_{n-1}(\sigma_{n-1} \cdot y) + \sigma_{n-2,k} \cdot p_{n-2}(\sigma_{n-2} \cdot y_{n-1}) + \cdots + p_k(\sigma_k \cdot y_{k+1}) \\ &= \sigma_{n,k} \cdot z + \sum_{i=k}^{n-1} \sigma_{i,k} \cdot p_i(\sigma_i \cdot y_{i+1}) \end{aligned}$$

where  $\sigma_{k+1,k} = \sigma_k$ . Moreover,  $H_i \circ \Lambda_i(w) = (\phi_i^{-1}(\sigma_i w), \sigma_i y, \bullet)$  for each  $k \leq i \leq n-1$ . Thus

$$\sigma_{n,i} \cdot y = \sigma_i \cdot y_{i+1} = y_i$$

Secondly, let us estimate  $\dot{z}^1 - \dot{z}^2 = \pi_z \circ \Psi_k^n(w^1) - \pi_z \circ \Psi_k^n(w^2)$  where  $w^j \in B(R^n F)$  for  $j = 1, 2$ . Recall the definition of  $q_i(y)$ , namely,  $\frac{d}{dy} p_i(y) = q_i(y)$ . By the above equation and mean value theorem, we obtain that

$$\dot{z}^1 - \dot{z}^2 = \pi_z \circ \Psi_k^n(w^1) - \pi_z \circ \Psi_k^n(w^2)$$

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<sup>3</sup>For notational compatibility, let  $\Psi_i^i(B) = B$ , that is,  $\Psi_i^i = \text{id}$  and let  $\sigma_{i,i} = 1$  for every  $i \in \mathbb{N}$ .

$$\begin{aligned}
&= \sigma_{n,k} \cdot (z^1 - z^2) + \sum_{i=k}^{n-1} \sigma_{i,k} \cdot [p_i(\sigma_i \cdot y_{i+1}^1) - p_i(\sigma_i \cdot y_{i+1}^2)] \\
&= \sigma_{n,k} \cdot (z^1 - z^2) + \sum_{i=k}^{n-1} \sigma_{i,k} \cdot [p_i(\sigma_{n,i} \cdot y^1) - p_i(\sigma_{n,i} \cdot y^2)] \\
&= \sigma_{n,k} \cdot (z^1 - z^2) + \sum_{i=k}^{n-1} \sigma_{i,k} \cdot q_i \circ (\sigma_{n,i} \cdot \bar{y}) \cdot \sigma_{n,i+1} \cdot (y^1 - y^2) \\
(5.3) \quad &= \sigma_{n,k} \cdot (z^1 - z^2) + \sigma_{n,k} \sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \cdot \bar{y}) \cdot (y^1 - y^2)
\end{aligned}$$

where  $\bar{y}$  is in the line segment between  $y^1$  and  $y^2$  which is contained in  $\pi_y \circ B(R^n F)$ . Moreover, by the expression of  $\Psi_k^n$ ,

$$\pi_z \circ \Psi_k^n(w) = \sigma_{n,k} [d_{n,k} y + z + R_k^n(y)].$$

Then

$$\begin{aligned}
\dot{z}^1 - \dot{z}^2 &= \pi_z \circ \Psi_k^n(w^1) - \pi_z \circ \Psi_k^n(w^2) \\
&= \sigma_{n,k} [d_{n,k} (y^1 - y^2) + (z_1 - z_2) + R_k^n(y^1) - R_k^n(y^2)] \\
(5.4) \quad &= \sigma_{n,k} \cdot (z^1 - z^2) + \sigma_{n,k} \cdot [d_{n,k} (y^1 - y^2) + R_k^n(y^1) - R_k^n(y^2)].
\end{aligned}$$

Hence, taking the limit as  $n \rightarrow \infty$ , we obtain

$$\sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \cdot \bar{y}) \cdot (y^1 - y^2) = d_{n,k} (y^1 - y^2) + R_k^n(y^1) - R_k^n(y^2)$$

of which convergence is exponentially fast. □

**Corollary 5.4.** *Let  $F \in \mathcal{I}(\bar{\varepsilon})$ . Then*

$$\sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) = d_{n,k} + (R_k^n)'(\pi_y(w))$$

for every  $w \in B(R^n F)$  and for each  $k < n$ . Moreover,

$$\lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) = d_{*,k}.$$

*Proof.* Let us compare the equation (5.3) and (5.4) below

$$\begin{aligned}
&\sum_{i=k}^{n-1} \sigma_{i,k} \cdot [p_i(\sigma_{n,i} \cdot y^1) - p_i(\sigma_{n,i} \cdot y^2)] = \sigma_{n,k} \cdot [d_{n,k} (y^1 - y^2) + R_k^n(y^1) - R_k^n(y^2)] \\
(5.5) \quad &\sum_{i=k}^{n-1} \sigma_{i,k} \cdot \frac{p_i(\sigma_{n,i} \cdot y^1) - p_i(\sigma_{n,i} \cdot y^2)}{y^1 - y^2} = \sigma_{n,k} \cdot \left[ d_{n,k} + \frac{R_k^n(y^1) - R_k^n(y^2)}{y^1 - y^2} \right].
\end{aligned}$$

Since both  $y^1$  and  $y^2$  are arbitrary, we may choose two points  $y$  and  $y + h$  instead of  $y^1$  and  $y^2$ . The differentiability of both  $p_i$  and  $R_k^n$  enable us to take the limit of (5.5) as  $h \rightarrow 0$ . Then

$$\sigma_{n,k} \cdot \sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \cdot y) = \sigma_{n,k} \cdot [d_{n,k} + (R_k^n)'(y)]$$

for every  $y \in \pi_y(B(R^n F))$ . Moreover,  $d_{n,k} \rightarrow d_{*,k}$  as  $n \rightarrow \infty$  super exponentially fast by Lemma 5.1 and  $(R_k^n)'$  converges to zero exponentially fast by Lemma 5.2. Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=k}^{n-1} q_i \circ (\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)) &= \lim_{n \rightarrow \infty} [d_{n,k} + (R_k^n)'(y)] \\ &= d_{*,k}. \end{aligned}$$

□

Let us collect the estimations of numbers and functions which are used in the following sections.

- (1)  $|t_{k+1,k}|$ ,  $|u_{k+1,k}|$  and  $|d_{k+1,k}|$  are  $O(\bar{\varepsilon}^{2^k})$ .
- (2)  $|u_{n,k}|$  and  $|d_{n,k}|$  are  $O(\bar{\varepsilon}^{2^k})$ .
- (3)  $\sigma_{n,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$  and  $\alpha_{n,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$  for  $k < n$  and for some  $0 < \rho < 1$ .
- (4)  $\|R_k^{k+1}\|$  is  $O(\bar{\varepsilon}^{2^k})$ .
- (5)  $\|R_k^n\|$  and  $\|(R_k^n)'\|$  are  $O(\sigma^{n-k}\bar{\varepsilon}^{2^k})$  by Lemma 5.2.

## 6. Unbounded geometry on critical Cantor set

**6.1. Boxing and bounded geometry.** Recall the *pieces*  $B_{\mathbf{w}}^n \equiv B_{\mathbf{w}}^n(F) = \Psi_{\mathbf{w}}^n(B)$  on the  $n^{\text{th}}$  level or  $n^{\text{th}}$  generation. The word,  $\mathbf{w} = (w_1 \dots w_n) \in W^n := \{v, c\}^n$  has length  $n$ . Recall that the map

$$\mathbf{w} = (w_1 \dots w_n) \mapsto \sum_{k=0}^{n-1} w_{k+1} 2^k$$

is one to one correspondence between words of length  $n$  and the additive group of numbers with base 2 mod  $2^n$ . Let the subset of critical Cantor set on each pieces be  $\mathcal{O}_{\mathbf{w}} \equiv B_{\mathbf{w}}^n \cap \mathcal{O}$ . Then by the definition of  $\mathcal{O}_{\mathbf{w}}$ , we have the following facts.

(1)

$$\mathcal{O}_F = \bigcup_{\mathbf{w} \in W^n} \mathcal{O}_{\mathbf{w}}$$

(2)  $F(B_{\mathbf{w}}^n) \subset B_{\mathbf{w}+1}^n$  for every  $\mathbf{w} = (w_1 \dots w_n) \in W^n$ .

(3)  $\text{diam}(B_{\mathbf{w}}^n) \leq C\sigma^n$  for some  $C > 0$  depending only on  $B$  and  $\bar{\varepsilon}$ .

Then we can define boxing of Cantor set. The notations in the following definitions are used in [HLM]. These definitions are adapted to three dimensional maps.

**Definition 6.1.** Let  $F \in \mathcal{I}(\bar{\varepsilon})$ . A collection of simply connected sets with interior  $\mathbf{B}^n = \{B_{\mathbf{w}}^n \in \text{Dom}(F) \mid \mathbf{w} \in W^n\}$  is called *boxing*<sup>4</sup> of  $\mathcal{O}_F$  if

- (1)  $\mathcal{O}_{\mathbf{w}} \subseteq B_{\mathbf{w}}^n$  for each  $\mathbf{w} \in W^n$ .
- (2)  $B_{\mathbf{w}}^n$  and  $B_{\mathbf{w}'}$  has disjoint closure if  $\mathbf{w} \neq \mathbf{w}'$ .
- (3)  $F(B_{\mathbf{w}}^n) \subset B_{\mathbf{w}+1}^n$  for every  $\mathbf{w} \in W^n$ .
- (4) Each element of  $\mathbf{B}^n$  is nested for each  $n$ , that is,

$$B_{\mathbf{w}\nu}^{n+1} \subset B_{\mathbf{w}}^n, \quad \mathbf{w} \in W^n, \quad \nu \in \{v, c\}.$$

On the above definition, the elements of boxing are just topological boxes. Denote  $\text{Dom}(F_{2d})$  by  $B_{2d}$  in order to distinguish the domain of three dimensional Hénon-like map from that of two dimensional one. Let the minimal distance between two boxes  $B_1, B_2$  be the infimum of distance between all elements of each boxes and express this distance to be  $\text{dist}_{\min}(B_1, B_2)$ .

**Definition 6.2.** The (given) boxing  $\mathbf{B}^n$  has the *bounded geometry* if

$$\begin{aligned} \text{dist}_{\min}(B_{\mathbf{w}v}^{n+1}, B_{\mathbf{w}c}^{n+1}) &\asymp \text{diam}(B_{\mathbf{w}\nu}^{n+1}) \quad \text{for } \nu \in \{v, c\} \\ \text{diam}(B_{\mathbf{w}}^n) &\asymp \text{diam}(B_{\mathbf{w}\nu}^{n+1}) \quad \text{for } \nu \in \{v, c\} \end{aligned}$$

for all  $\mathbf{w} \in W^n$  and for all  $n \geq 0$ .

Moreover, if the boxing has bounded geometry, then we just call  $\mathcal{O}_F$  has bounded geometry. If any given boxing does not have bounded geometry, then we call  $\mathcal{O}_F$  has *unbounded geometry*.

**6.2. Horizontal overlap of two adjacent boxes.** The proof of unbounded geometry of the Cantor set requires to compare diameter of boxes and the minimal distance of two adjacent boxes in the boxing. In order to compare these quantities, we would use the maps,  $\Psi_k^n(w)$  and  $F_k(w)$  with the two points  $w_1 = (x_1, y_1, z_1)$  and  $w_2 = (x_2, y_2, z_2)$  in  $F_n(B)$ . Let us each successive image of  $w_j$  under  $\Psi_k^n(w)$  and  $F_k(w)$  be  $\dot{w}_j$ ,  $\ddot{w}_j$  and  $\ddot{\dot{w}}_j$  for  $j = 1, 2$ .

$$w_j \xrightarrow{\Psi_k^n} \dot{w}_j \xrightarrow{F_k} \ddot{w}_j \xrightarrow{\Psi_0^k} \ddot{\dot{w}}_j$$

For example,  $\dot{w}_j = \Psi_k^n(w_j)$  and  $\dot{w}_j = (\dot{x}_j, \dot{y}_j, \dot{z}_j)$  for  $j = 1, 2$ . Let  $S_1$  and  $S_2$  be the (path) connected set on  $\mathbb{R}^3$ . If  $\pi_x(\overline{S_1}) \cap \pi_x(\overline{S_2})$  contains at least two points, then this intersection is called the *x-axis overlap* or *horizontal overlap* of  $S_1$  and  $S_2$ . We say  $S_1$  *overlaps*  $S_2$  on the *x-axis* or *horizontally*. Recall  $\sigma$  is the linear scaling of  $F_*$ , the fixed point of renormalization operator and  $\sigma_k = \sigma(1 + O(\rho^k))$  for each  $k \in \mathbb{N}$ .

Recall the map  $\Psi_k^n$  from  $B(R^n F)$  to  $B_{\mathbf{v}}^n(R^k F)$  where  $\mathbf{v} = v^{n-k} \in W^{n-k}$ .

$$\Psi_k^n(w) = \begin{pmatrix} 1 & t_{n,k} & u_{n,k} \\ & 1 & \\ d_{n,k} & & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,k} & & \\ & \sigma_{n,k} & \\ & & \sigma_{n,k} \end{pmatrix} \begin{pmatrix} x + S_k^n(w) \\ y \\ z + R_k^n(y) \end{pmatrix}$$

---

<sup>4</sup>Element  $B_{\mathbf{w}}^n$  in Definition 6.1 is defined topologically. In other words, this definition does not require that  $B_{\mathbf{w}}^n \in \mathbf{B}^n$  is  $\Psi_{\mathbf{w}}^n(B)$ .

where  $\alpha_{n,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$  and  $\sigma_{n,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$ . Thus for any  $w \in B(R^n F)$  we have the following equation

$$\pi_x \circ \Psi_k^n(w) = \alpha_{n,k}(x + S_k^n(w)) + \sigma_{n,k}(t_{n,k}y + u_{n,k}(z + R_k^n(y))).$$

Let us find the sufficient condition of horizontal overlapping. Horizontal overlapping means that there exist two points  $w_1 \in B_v^1(R^n F)$  and  $w_2 \in B_c^1(R^n F)$  satisfying the equation

$$\pi_x \circ \Psi_k^n(w_1) - \pi_x \circ \Psi_k^n(w_2) = 0.$$

Equivalently,

$$(6.1) \quad \begin{aligned} & \alpha_{n,k} \left[ (x_1 + S_k^n(w_1)) - (x_2 + S_k^n(w_2)) \right] \\ & + \sigma_{n,k} \left[ t_{n,k}(y_1 - y_2) + u_{n,k} \{ z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2) \} \right] = 0. \end{aligned}$$

Recall that  $x + S_k^n(w) = v_*(x) + O(\varepsilon^{2^k} + \rho^{n-k})$  for some  $0 < \rho < 1$ . Since the universal map  $v_*(x)$  is a diffeomorphism and  $|x_1 - x_2| = O(1)$ , we have the estimation by mean value theorem

$$|x_1 + S_k^n(w_1) - (x_2 + S_k^n(w_2))| = O(1).$$

Recall  $\tau_i$  the tip of  $F_i$  for  $i \in \mathbb{N}$ .

**Proposition 6.1.** *Let  $F \in \mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Let  $b_1 = b_F/b_2$  where  $b_F$  is the average Jacobian of  $F$  and  $b_2$  is the universal number defined in Proposition 4.3. Then*

$$b_1^{2^k} \asymp t_{n,k}$$

for every  $k + A < n$  where  $A$  is the big enough number depending only on  $b_1$  and  $\bar{\varepsilon}$ .

*Proof.* Applying Lemma 4.2 to  $\delta_k$ , we see

$$\partial_x \delta_n(w) = \partial_x \delta_k \circ \Psi_{k,c}^n(w) - \sum_{i=k}^{n-1} q_i \circ (\pi_x \circ \Psi_{i,c}^n(w))$$

for  $k < n$ . Let us take  $w = c_{F_n}$  which is the critical point of  $F_n$ . Then

$$\partial_x \delta_n(c_{F_n}) = \partial_x \delta_k(c_{F_k}) - \sum_{i=k}^{n-1} q_i \circ (\pi_x(c_{F_i})) = \partial_x \delta_k(c_{F_k}) - \sum_{i=k}^{n-1} q_i \circ (\pi_y(\tau_i))$$

where  $c_{F_i}$  is the critical point of  $F_i$  for  $k \leq i \leq n-1$  because  $F_i(c_{F_i}) = \tau_i$ . Moreover, by Proposition 4.3,  $\partial_z \delta_k = b_2^{2^k}(1 + O(\rho^n))$  for some positive  $\rho < 1$ . The fact that  $F \in \mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$  implies that  $\partial_y \delta_k(F_k(w)) = -\partial_z \delta_k(F_k(w)) \cdot \partial_x \delta_k(w)$ . Then  $\text{Jac } F_k(\tau_k)$  as follows

$$\begin{aligned} \text{Jac } F_k(\tau_k) &= \partial_y \varepsilon_k(\tau_k) \cdot \partial_z \delta_k(\tau_k) - \partial_z \varepsilon_k(\tau_k) \cdot \partial_y \delta_k(\tau_k) \\ &= [\partial_y \varepsilon_k(\tau_k) + \partial_z \varepsilon_k(\tau_k) \cdot \partial_x \delta_k(c_{F_k})] \cdot \partial_z \delta_k(\tau_k) \\ &= [\partial_y \varepsilon_k(\tau_k) + \partial_z \varepsilon_k(\tau_k) \cdot \sum_{i=k}^{n-1} q_i \circ (\pi_y(\tau_i))] \cdot b_2^{2^k}(1 + O(\rho^k)) \\ &\quad + \partial_z \varepsilon_k(\tau_k) \cdot \partial_x \delta_n(c_{F_n}) \cdot b_2^{2^k}(1 + O(\rho^k)). \end{aligned}$$

By universality theorem,  $\text{Jac } F_k(\tau_k) = b_F^{2^k} a(\pi_x(\tau_k))(1 + O(\rho^k))$  and by the definition of  $b_1$ , we obtain that

(6.2)

$$\partial_y \varepsilon_k(\tau_k) + \partial_z \varepsilon_k(\tau_k) \cdot \sum_{i=k}^{n-1} q_i \circ (\pi_y(\tau_i)) + \partial_z \varepsilon_k(\tau_k) \cdot \partial_x \delta_n(c_{F_n}) = b_1^{2^k} a(\pi_x(\tau_k))(1 + O(\rho^k)).$$

Observe that  $\|\partial_z \varepsilon_k \cdot \partial_x \delta_n\| = O(\bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n})$  and  $k < n$ . Let us find the sufficient condition satisfying  $\bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n} \lesssim b_1^{2^k}$ . If  $b_1 \geq \bar{\varepsilon}^2$ , then  $\bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n} \leq b_1^{2^k}$  for  $n > k$ . Assume that  $b_1 < \bar{\varepsilon}^2 \ll 1$ . Thus

$$\begin{aligned} \bar{\varepsilon}^{2^n} \bar{\varepsilon}^{2^k} \lesssim b_1^{2^k} &\iff (2^n + 2^k) \log \bar{\varepsilon} \lesssim 2^k \log b_1 \\ &\iff 2^n \geq 2^k \left( \frac{\log b_1}{\log \bar{\varepsilon}} - 1 \right) + C_0 \end{aligned}$$

for some positive  $C_0 > 0$ . Define  $A$  as follows

$$(6.3) \quad A = \begin{cases} 0 & \text{where } b_1 \geq \bar{\varepsilon}^2 \\ C_1 \log_2 \left( \frac{\log b_1}{\log \bar{\varepsilon}} - 1 \right) & \text{where } b_1 < \bar{\varepsilon}^2 \end{cases}$$

Thus if  $n \geq k + A$ , then  $\bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n} \lesssim b_1^{2^k}$ . Recall that  $H_k^{-1} \circ \Lambda_k^{-1} = \Psi_k^{k+1}$ . Thus let us compare each components of derivatives,  $D(\Psi_k^{k+1})^{-1}(\tau_k) = (D_k^{k+1})^{-1} = D(\Lambda_k \circ H_k)(\tau_k)$ . Then comparison of the second and the third column of the matrices shows that

$$\begin{aligned} \frac{\alpha_{k+1,k}}{\sigma_{k+1,k}} \cdot \partial_y \varepsilon_k(\tau_k) &= t_{k+1,k} - u_{k+1,k} d_{k+1,k} \\ \frac{\alpha_{k+1,k}}{\sigma_{k+1,k}} \cdot \partial_z \varepsilon_k(\tau_k) &= u_{k+1,k}, \quad q_i \circ (\pi_y(\tau_i)) = d_{i+1,i}. \end{aligned}$$

The equation,  $d_{n,k+1} = \sum_{i=k+1}^n d_{i+1,i}$  holds by Lemma 5.1. Then

$$\begin{aligned} t_{k+1,k} + u_{k+1,k} \cdot d_{n,k+1} &= t_{k+1,k} - u_{k+1,k} \cdot d_{k+1,k} + u_{k+1,k} \cdot d_{n,k} \\ &= t_{k+1,k} - u_{k+1,k} \cdot d_{k+1,k} + u_{k+1,k} \sum_{i=k}^{n-1} d_{i+1,i} \\ &= \frac{\alpha_{k+1,k}}{\sigma_{k+1,k}} \left[ \partial_y \varepsilon_k(\tau_k) + \partial_z \varepsilon_k(\tau_k) \sum_{i=k}^{n-1} q_i \circ (\pi_y(\tau_i)) \right] \\ &= b_1^{2^k} \cdot [\sigma \cdot a(\pi_x(\tau_k))(1 + O(\rho^k)) + C_1] \end{aligned}$$

for some  $C_1 > 0$  where  $n \geq k + A$ . Hence, by Lemma 5.1 again,  $t_{n,k}$  is as follows

$$\begin{aligned} t_{n,k} &\asymp \sum_{i=k}^{n-1} \sigma^{i-k} [t_{i+1,i} + u_{i+1,i} \cdot d_{n,i+1}] (1 + O(\rho^k)) \\ &\asymp t_{k+1,k} + u_{k+1,k} \cdot d_{n,k+1} \\ &\asymp b_1^{2^k}. \end{aligned}$$

□

Let us choose two points  $w'_1$  and  $w'_2$  in  $F_n(B)$ . In particular, we may assume that  $w_j \in \mathcal{O}_{R^n F}$  where  $\pi_z(w_j) = z_j$  for  $j = 1, 2$ . Let  $w'_1$  and  $w'_2$  be the pre-image of  $w_1$  and  $w_2$  respectively. Then

$$(6.4) \quad |z_1 - z_2| = |\delta_n(w'_1) - \delta_n(w'_2)| \leq C \|D\delta_n\| \cdot \|w'_1 - w'_2\| = O(\bar{\varepsilon}^{2^n})$$

for some  $C > 0$ . Thus  $|z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$ .

**Corollary 6.2.** *Let  $F \in \mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Suppose that  $\Psi_k^n(B_v^{n+1})$  overlaps  $\Psi_k^n(B_c^{n+1})$  on  $x$ -axis. In particular,  $\pi_x \circ \Psi_k^n(w_1) = \pi_x \circ \Psi_k^n(w_2)$  where  $w_1 \in B_v^{n+1} \cap \mathcal{O}_{R^n F}$  and  $w_2 \in B_c^{n+1} \cap \mathcal{O}_{R^n F}$ . Suppose also that  $k$  and  $n$  are big enough and  $n \geq k + A$  where  $A$  is the number defined in Proposition 6.1. Then*

$$\sigma^{n-k} \asymp b_1^{2^k}$$

for every big enough  $k$ .

*Proof.* Recall the following equation

$$\pi_x \circ \Psi_k^n(w) = \alpha_{n,k} [x + S_k^n(w)] + \sigma_{n,k} [t_{n,k} y + u_{n,k} (z + R_k^n(y))].$$

Recall  $x + S_k^n(w) = v_*(x) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k})$  where  $v_*(x)$  is a diffeomorphism and

$$|v_*(x_1) - v_*(x_2)| = |v'_*(\bar{x}) \cdot (x_1 - x_2)| \geq C_0 > 0$$

where  $\bar{x}$  is in the line segment between  $x_1$  and  $x_2$ . Thus

$$\begin{aligned} \dot{x}_1 - \dot{x}_2 &= \alpha_{n,k} [(x_1 + S_k^n(w_1)) - (x_2 + S_k^n(w_2))] \\ &\quad + \sigma_{n,k} [t_{n,k}(y_1 - y_2) + u_{n,k} \{z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)\}] \\ &= \alpha_{n,k} [v'_*(\bar{x}) \cdot (x_1 - x_2) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k})] \\ &\quad + \sigma_{n,k} [t_{n,k}(y_1 - y_2) + u_{n,k} \{z_1 - z_2 + (R_k^n)'(\bar{y}) \cdot (y_1 - y_2)\}]. \end{aligned}$$

Then by Proposition 6.1 and the estimations in the end of Section 6.2, we obtain that

$$\begin{aligned} |\dot{x}_1 - \dot{x}_2| &= \left| C_3 \sigma^{2(n-k)} + \sigma^{n-k} [C_4 b_1^{2^k} + C_5 \bar{\varepsilon}^{2^k} (\bar{\varepsilon}^{2^n} + \sigma^{n-k} \bar{\varepsilon}^{2^k})] \right| \\ &= \left| \sigma^{2(n-k)} [C_3 + C_4 \bar{\varepsilon}^{2^{k+1}}] + \sigma^{n-k} [C_4 b_1^{2^k} + C_5 \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}] \right| \\ (6.5) \quad &\leq C_5 \sigma^{2(n-k)} + C_6 \sigma^{n-k} [b_1^{2^k} + \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}] \end{aligned}$$

for some constants  $C_3, C_4$ , and  $C_5$ , which do not have to be positive. Let us take big enough  $n$  such that the condition  $b_1^{2^k} \gtrsim \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}$  is satisfied by Proposition 6.1. However,  $\dot{x}_1 - \dot{x}_2 = 0$  is the horizontal overlapping assumption. Hence,

$$\sigma^{2(n-k)} \asymp \sigma^{n-k} b_1^{2^k}.$$

□



**6.3. Unbounded geometry on the critical Cantor set.** Let us assume that the  $x$ -axis overlapping of two boxes,  $\Psi_{k,\mathbf{v}}^n(B_v^{n+1})$  and  $\Psi_{k,\mathbf{v}}^n(B_c^{n+1})$ . Under this assumption, we can measure the upper bound the minimal distances of two adjacent boxes  $\text{dist}_{\min}(B_{\mathbf{w}v}^n, B_{\mathbf{w}c}^n)$ , where  $B_{\mathbf{w}v}^n$  and  $B_{\mathbf{w}c}^n$  are the image of  $B_v^{n+1}$  and  $B_c^{n+1}$  under  $\Psi_{0,\mathbf{v}}^k \circ F_k \circ \Psi_{k,\mathbf{v}}^n$  respectively. Compare this minimal distance with the lower bound of the diameter of the above boxes. Then Cantor attractor has unbounded geometry. Moreover, this result is only related to the universal constant  $b_1$  (Theorem 6.7).

**Lemma 6.3.** *Let  $F$  be the Hénon-like diffeomorphism in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Suppose that  $B_{\mathbf{v}v}^n(R^k F)$  overlaps  $B_{\mathbf{v}c}^n(R^k F)$  on the  $x$ -axis where the word  $\mathbf{v} = v^{n-k} \in W^{n-k}$ . Then*

$$\text{dist}_{\min}(B_{\mathbf{w}v}^n, B_{\mathbf{w}c}^n) \leq C \left[ \sigma^{2k} \sigma^{n-k} b_1^{2k} + \sigma^{2k} \sigma^{2(n-k)} \bar{\varepsilon}^{2k} + \sigma^k \sigma^{2(n-k)} b_2^{2k} \right]$$

where  $\mathbf{w} = v^k c v^{n-k-1} \in W^n$  for some  $C > 0$  and sufficiently big  $k$ .

*Proof.* Recall the map  $\Psi_k^n$  from  $B(R^n F)$  to  $B_{\mathbf{v}}^{n-k}(R^k F)$

$$\Psi_k^n(w) = \begin{pmatrix} 1 & t_{n,k} & u_{n,k} \\ & 1 & \\ & d_{n,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,k} & & \\ & \sigma_{n,k} & \\ & & \sigma_{n,k} \end{pmatrix} \begin{pmatrix} x + S_k^n(w) \\ y \\ z + R_k^n(y) \end{pmatrix}.$$

where  $\mathbf{v} = v^{n-k} \in W^{n-k}$ . Let us choose two different points as follows

$$w_1 = (x_1, y_1, z_1) \in B_v^1(R^n F) \cap \mathcal{O}_{R^n F}, \quad w_2 = (x_2, y_2, z_2) \in B_c^1(R^n F) \cap \mathcal{O}_{R^n F}.$$

Then by the above expression of  $\Psi_k^n$  and overlapping on the  $x$ -axis, we may assume the following equations

$$\begin{aligned} \dot{x}_1 - \dot{x}_2 &= 0 \\ \dot{y}_1 - \dot{y}_2 &= \sigma_{n,k}(y_1 - y_2) \\ \dot{z}_1 - \dot{z}_2 &= \sigma_{n,k} [d_{n,k}(y_1 - y_2) + z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)]. \end{aligned}$$

Moreover, definitions of  $F_k$  and  $\Psi_{0,\mathbf{v}}^k$  implies that

$$\ddot{y}_1 - \ddot{y}_2 = \sigma_{k,0} \cdot (\ddot{y}_1 - \ddot{y}_2) = \sigma_{k,0} \cdot (\dot{x}_1 - \dot{x}_2) = 0.$$

By mean value theorem and the fact that  $(\ddot{x}_j, \ddot{y}_j, \ddot{z}_j) = R^k F(\dot{x}_j, \dot{y}_j, \dot{z}_j)$  for  $j = 1, 2$ , we obtain the following equations

$$\begin{aligned} \ddot{x}_1 - \ddot{x}_2 &= f_k(\dot{x}_1) - \varepsilon_k(\dot{w}_1) - [f_k(\dot{x}_2) - \varepsilon_k(\dot{w}_2)] \\ &= -\varepsilon_k(\dot{w}_1) + \varepsilon_k(\dot{w}_2) \\ &= -\partial_y \varepsilon_k(\eta) \cdot (\dot{y}_1 - \dot{y}_2) - \partial_z \varepsilon_k(\eta) \cdot (\dot{z}_1 - \dot{z}_2) \\ &= -\partial_y \varepsilon_k(\eta) \cdot \sigma_{n,k}(y_1 - y_2) \\ &\quad - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k} [d_{n,k}(y_1 - y_2) + z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)] \\ (*) &= -\partial_y \varepsilon_k(\eta) \cdot \sigma_{n,k}(y_1 - y_2) - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k} \sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \bar{y}) \cdot (y_1 - y_2) \\ &\quad - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k}(z_1 - z_2) \end{aligned}$$

$$(6.6) \quad = - \left[ \partial_y \varepsilon_k(\eta) + \partial_z \varepsilon_k(\eta) \cdot \sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \bar{y}) \right] \cdot \sigma_{n,k}(y_1 - y_2) - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k}(z_1 - z_2)$$

where  $\eta$  is some point in the line segment between  $\dot{w}_1$  and  $\dot{w}_2$  in  $\Psi_k^n(B)$  and  $\bar{y}$  is in the line segment between  $y_1$  and  $y_2$ . The second last equation (\*) is involved with Proposition 5.3. Recall that  $|y_1 - y_2| \asymp 1$  and  $|z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$  because every point in the critical Cantor set,  $\mathcal{O}_{F_n}$  has its inverse image under  $F_n$ . Thus by Lemma 4.6, we obtain that

$$(6.7) \quad |\ddot{x}_1 - \ddot{x}_2| \leq C_1 \sigma^{n-k} [b_1^{2^k} + \bar{\varepsilon}^{2^k} \sigma^{n-k} + \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}].$$

Similarly, we have

$$\begin{aligned} \ddot{z}_1 - \ddot{z}_2 &= \delta_k(\dot{w}_1) - \delta_k(\dot{w}_2) \\ &= \partial_y \delta_k(\zeta) \cdot (\dot{y}_1 - \dot{y}_2) + \partial_z \delta_k(\zeta) \cdot (\dot{z}_1 - \dot{z}_2) \\ (6.8) \quad &= \partial_y \delta_k(\zeta) \cdot \sigma_{n,k}(y_1 - y_2) \\ &\quad + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k} [d_{n,k}(y_1 - y_2) + z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)] \\ &= \partial_y \delta_k(\zeta) \cdot \sigma_{n,k}(y_1 - y_2) + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k} \sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \bar{y}) \cdot (y_1 - y_2) \\ &\quad + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k}(z_1 - z_2) \\ &= \left[ \partial_y \delta_k(\zeta) + \partial_z \delta_k(\zeta) \sum_{i=k}^{n-1} q_i \circ (\sigma_{n,i} \bar{y}) \right] \cdot \sigma_{n,k}(y_1 - y_2) + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k}(z_1 - z_2) \end{aligned}$$

where  $\zeta$  is some point in the line segment between  $\dot{w}_1$  and  $\dot{w}_2$  in  $\Psi_k^n(B)$ . By Lemma 4.4, the upper bounds of  $|\ddot{z}_1 - \ddot{z}_2|$  is

$$(6.9) \quad |\ddot{z}_1 - \ddot{z}_2| \leq C_2 \sigma^{n-k} [\sigma^{n-k} b_2^{2^k} + b_2^{2^k} \bar{\varepsilon}^{2^n}].$$

Recall

$$\pi_x \circ \Psi_k^n(w) = \alpha_{n,k} [x + S_k^n(w)] + \sigma_{n,k} [t_{n,k} y + u_{n,k}(z + R_k^n(y))].$$

Then the fact that  $\ddot{y}_1 - \ddot{y}_2 = 0$  implies that

$$\begin{aligned} \ddot{x}_1 - \ddot{x}_2 &= \pi_x \circ \Psi_0^k(\ddot{w}_1) - \pi_x \circ \Psi_0^k(\ddot{w}_2) \\ &= \alpha_{k,0} [(\ddot{x}_1 + S_0^k(\ddot{w}_1)) - (\ddot{x}_2 + S_0^k(\ddot{w}_2))] \\ &\quad + \sigma_{k,0} [t_{k,0}(\ddot{y}_1 - \ddot{y}_2) + u_{k,0}(\ddot{z}_1 - \ddot{z}_2 + R_0^k(\ddot{y}_1) - R_0^k(\ddot{y}_2))] \\ &= \alpha_{k,0} [v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k)] (\ddot{x}_1 - \ddot{x}_2) + \sigma_{k,0} \cdot u_{k,0}(\ddot{z}_1 - \ddot{z}_2) \end{aligned}$$

where  $\bar{x}$  is some point in the line segment between  $\ddot{x}_1$  and  $\ddot{x}_2$ . Moreover,

$$\begin{aligned} \ddot{z}_1 - \ddot{z}_2 &= \pi_z \circ \Psi_0^k(\ddot{w}_1) - \pi_z \circ \Psi_0^k(\ddot{w}_2) \\ &= \sigma_{k,0}(\ddot{z}_1 - \ddot{z}_2) + \sigma_{k,0} [d_{k,0}(\ddot{y}_1 - \ddot{y}_2) + R_k^n(\ddot{y}_1) - R_k^n(\ddot{y}_2)] \\ &= \sigma_{k,0}(\ddot{z}_1 - \ddot{z}_2). \end{aligned}$$

Let us apply the estimations in (6.7) and (6.9) to  $\ddot{x}_1 - \ddot{x}_2$  and  $\ddot{z}_1 - \ddot{z}_2$ . Then the minimal distance is bounded above as follows

$$\begin{aligned}
\text{dist}_{\min}(B_{\mathbf{w}v}^n, B_{\mathbf{w}c}^n) &\leq |\ddot{x}_1 - \ddot{x}_2| + |\ddot{z}_1 - \ddot{z}_2| \\
&\leq [\sigma^{2k} \cdot |\ddot{x}_1 - \ddot{x}_2| \cdot v_*(\bar{x}) + \sigma^k \cdot (1 + u_{k,0}) |\ddot{z}_1 - \ddot{z}_2|] (1 + O(\rho^k)) \\
&\leq C_3 \sigma^{2k} \sigma^{n-k} [b_1^{2k} + \bar{\varepsilon}^{2k} \sigma^{n-k} + \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n}] + C_4 \sigma^k \sigma^{n-k} [\sigma^{n-k} b_2^{2k} + b_2^{2k} \bar{\varepsilon}^{2n}] \\
(6.10) \quad &\leq C_3 [\sigma^{2k} \sigma^{n-k} b_1^{2k} + \sigma^{2k} \sigma^{2(n-k)} \bar{\varepsilon}^{2k}] + C_4 \sigma^k \sigma^{2(n-k)} b_2^{2k}
\end{aligned}$$

for some positive numbers  $C_3$  and  $C_4$ . Hence, the estimation (6.10) is refined as follows

$$\begin{aligned}
(6.11) \quad \text{dist}_{\min}(B_{\mathbf{w}v}^n, B_{\mathbf{w}c}^n) &\leq |\ddot{x}_1 - \ddot{x}_2| + |\ddot{z}_1 - \ddot{z}_2| \\
&\leq C [\sigma^{2k} \sigma^{n-k} b_1^{2k} + \sigma^{2k} \sigma^{2(n-k)} \bar{\varepsilon}^{2k} + \sigma^k \sigma^{2(n-k)} b_2^{2k}]
\end{aligned}$$

for some  $C > 0$ . □

**Lemma 6.4.** *Let  $F \in \mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Then*

$$\text{diam}(B_{\mathbf{w}v}^n) \geq |C_1 \sigma^k \sigma^{2(n-k)} - C_2 \sigma^k \sigma^{n-k} b_1^{2k}|$$

where  $\mathbf{w} = v^k c v^{n-k-1} \in W^n$  and  $n$  is big enough satisfying  $n \geq k + A$  for  $A$  defined in Proposition 6.1.

*Proof.* Let us choose two points

$$w_j = (x_j, y_j, z_j) \in B_v^1(R^n F) \cap \mathcal{O}_{R^n F}$$

for  $j = 1, 2$  satisfying  $|x_1 - x_2| \asymp 1$  and  $|y_1 - y_2| = O(1)$ . Thus we may assume that  $|z_1 - z_2| = O(\bar{\varepsilon}^{2n})$  by the equation (6.4). Recall that the box,  $B_{\mathbf{w}v}^n$  is  $\Psi_{0,\mathbf{w}}^n(B_v^1(R^n F))$  and that the diameter of the box  $B_{\mathbf{w}v}^n$  is greater than the distance between any two points in  $B_{\mathbf{w}v}^n$ . Let  $\ddot{w}_j = \Psi_{0,\mathbf{v}}^k \circ F_k \circ \Psi_{k,\mathbf{v}}^n(w_j)$ ,  $\Psi_{0,\mathbf{v}}^k(w_j) = \dot{w}_j$ , and  $F_k(\dot{w}_j) = \ddot{w}_j$  for  $j = 1, 2$ . Then

$$\text{diam}(B_v^1) = \sup \{ |w_1 - w_2| \mid w_1, w_2 \in B_v^1 \} \asymp 1.$$

We may assume that  $|x_1 - x_2| \asymp 1$  and  $|y_1 - y_2| \asymp 1$  by the appropriate choice of  $w_1$  and  $w_2$ . The definition of  $F_k$  and  $\Psi_0^k$  implies that

$$\begin{aligned}
\text{diam}(B_{\mathbf{w}v}^n) &\geq |\ddot{w}_1 - \ddot{w}_2| \geq |\ddot{y}_1 - \ddot{y}_2| \\
&= |\sigma_{k,0}(\ddot{y}_1 - \ddot{y}_2)| \\
&= |\sigma_{k,0}(\dot{x}_1 - \dot{x}_2)| \\
&= |\sigma_{k,0}[\pi_x \circ \Psi_k^n(w_1) - \pi_x \circ \Psi_k^n(w_2)]|
\end{aligned}$$

for any two points  $\ddot{w}_1, \ddot{w}_2 \in B_{\mathbf{w}v}^n$ . Recall the equation

$$\pi_x \circ \Psi_k^n(w) = \alpha_{n,k}[x + S_k^n(w)] + \sigma_{n,k}[t_{n,k} y + u_{n,k}(z + R_k^n(y))].$$

and recall  $x + S_k^n(w) = v_*(x) + O(\bar{\varepsilon}^{2k} + \rho^{n-k})$ . Thus

$$|v_*(x_1) - v_*(x_2)| = |v'_*(\bar{x}) \cdot (x_1 - x_2)| \geq C_0 > 0$$

where  $v_*(x)$  is a diffeomorphism and  $\bar{x}$  is in the line segment between  $x_1$  and  $x_2$ . Thus

$$\begin{aligned}\dot{x}_1 - \dot{x}_2 &= \alpha_{n,k} \left[ (x_1 + S_k^n(w_1)) - (x_2 + S_k^n(w_2)) \right] \\ &\quad + \sigma_{n,k} \left[ t_{n,k}(y_1 - y_2) + u_{n,k} \{ z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2) \} \right] \\ &= \alpha_{n,k} \left[ v'_*(\bar{x}) \cdot (x_1 - x_2) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k}) \right] \\ &\quad + \sigma_{n,k} \left[ t_{n,k}(y_1 - y_2) + u_{n,k} \{ z_1 - z_2 + (R_k^n)'(\bar{y}) \cdot (y_1 - y_2) \} \right]\end{aligned}$$

Then by Proposition 6.1 and the estimations in the end of Section 6.2, we obtain that

$$\begin{aligned}(6.12) \quad |\dot{x}_1 - \dot{x}_2| &= \left| C_3 \sigma^{2(n-k)} + \sigma^{n-k} [C_4 b_1^{2^k} + C_5 \bar{\varepsilon}^{2^k} (\bar{\varepsilon}^{2^n} + \sigma^{n-k} \bar{\varepsilon}^{2^k})] \right| \\ &= \left| \sigma^{2(n-k)} [C_3 + C_4 \bar{\varepsilon}^{2^{k+1}}] + \sigma^{n-k} [C_4 b_1^{2^k} + C_5 \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}] \right|\end{aligned}$$

for some constants  $C_3$ ,  $C_4$ , and  $C_5$  which do not have to be positive. Let us take big enough  $n$  such that the condition  $b_1^{2^k} \gtrsim \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}$  is satisfied.

$$n \geq k + A$$

where  $A$  is the number depending only on  $\bar{\varepsilon}$  and  $b_1$  in Proposition 6.1. Hence,

$$\text{diam}(B_{\mathbf{w}}^n) \geq |\ddot{y}_1 - \ddot{y}_2| \geq |\sigma_{k,0}(\dot{x}_1 - \dot{x}_2)| \geq |C_1 \sigma^k \sigma^{2(n-k)} - C_2 \sigma^k \sigma^{n-k} b_1^{2^k}|$$

where  $\mathbf{w} = v^k_C v^{n-k-1} \in W^n$  for some positive  $C_1$  and  $C_2$ . □

*Remark 6.1.* In the above lemma, we may choose two points  $w_1$  and  $w_2$  which maximize  $|\dot{x}_1 - \dot{x}_2|$ . Thus we may assume that

$$\text{diam}(B_{\mathbf{w}}^n) \geq \max \{ C_1 \sigma^k \sigma^{2(n-k)}, C_2 \sigma^k \sigma^{n-k} b_1^{2^k} \}$$

with appropriate positive constants  $C_1$  and  $C_2$ .

Horizontal (or  $x$ -axis) overlap is only related to the  $x$ -coordinates of points  $\dot{w}_j \equiv \pi_x \circ \Psi_k^n(w_j)$  for  $j = 1, 2$  where  $w_1 \in B_v^1(R^n F) \cap \mathcal{O}_{R^n F}$  and  $w_2 \in B_c^1(R^n F) \cap \mathcal{O}_{R^n F}$ . Recall that  $b_2$  is the number defined in Proposition 4.3 and  $b_1$  is defined by the equation,  $b_1 b_2 = b_F$  where  $b_F$  is the average Jacobian of  $F$ . Recall that  $b_1$  is also another universal constant by Lemma 4.5.

**Proposition 6.5.** *Let  $F_{b_1}$  be an element of parametrized space in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . If  $\sigma^{n-k} \asymp b_1^{2^k}$  for infinitely many  $k$  and  $n$ , then there exists  $b_1$  for  $F_{b_1}$  such that  $B_{\mathbf{v}v}^n(R^k F)$  overlaps  $B_{\mathbf{v}c}^n(R^k F)$  on the  $x$ -axis where the word  $\mathbf{v} = v^{n-k} \in W^{n-k}$ . Furthermore,  $F$  has no bounded geometry of  $\mathcal{O}_F$ .*

*Proof.* Let us choose the two points

$$w_1 = (x_1, y_1, z_1) \in B_v^1(R^n F) \cap \mathcal{O}_{R^n F}, \quad w_2 = (x_2, y_2, z_2) \in B_c^1(R^n F) \cap \mathcal{O}_{R^n F}$$

such that  $|x_1 - x_2| \asymp 1$ ,  $|y_1 - y_2| \asymp 1$ . Recall  $|z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$ . Let  $\dot{w}_j = (\dot{x}_j, \dot{y}_j, \dot{z}_j)$  be  $\Psi_k^n(w_j)$  for  $j = 1, 2$ . Thus

$$(6.13) \quad \begin{aligned} \dot{x}_1 - \dot{x}_2 = & \alpha_{n,k} \left[ (x_1 + S_k^n(w_1)) - (x_2 + S_k^n(w_2)) \right] \\ & + \sigma_{n,k} \left[ t_{n,k}(y_1 - y_2) + u_{n,k} \{ z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2) \} \right]. \end{aligned}$$

Recall that  $\alpha_{n,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$ ,  $\sigma_{n,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$  and  $x + S_k^n(w) = v_*(x) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k})$ . Since  $v_*$  is a diffeomorphism and  $|x_1 - x_2| \asymp 1$ ,  $|v_*(x_1) - v_*(x_2)| \asymp 1$  by mean value theorem. Moreover, Proposition 6.1 implies that

$$b_1^{2^k} \asymp t_{n,k}.$$

In addition to the above estimation, the fact that  $\|(R_k^n)'\| = O(\sigma^{n-k}\bar{\varepsilon}^{2^k})$  and the estimation in (6.5) implies that

$$\begin{aligned} |u_{n,k} [z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)]| & \leq |u_{n,k} \cdot (z_1 - z_2)| + |(R_k^n)'(\bar{y}) \cdot (y_1 - y_2)| \\ & = O(\bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}) + O(\sigma^{n-k} \bar{\varepsilon}^{2^k}). \end{aligned}$$

If  $n \geq k + A$ , then we express the equation (6.13) as follows

$$\dot{x}_1 - \dot{x}_2 = \sigma^{2(n-k)} [v_*(x_1) - v_*(x_2)] \cdot [1 + r_{n,k} b_1^{2^k} (-\sigma)^{-(n-k)}] (1 + O(\rho^k))$$

where  $\frac{1}{r} \leq r_{n,k} \leq r$  for some number  $r$  depends uniformly on  $b_1$ . Let us take  $n$  such that

$$\sigma^{n-k} \asymp b_1^{2^k}.$$

Then we may assume that  $B_{\mathbf{v}_v}^{n-k}(R^k F)$  overlaps  $B_{\mathbf{v}_c}^{n-k}(R^k F)$  on the  $x$ -axis where  $\mathbf{v} = v^{n-k-1} \in W^{n-k-1}$  for infinitely many big enough  $n - k$ . Let us compare the distance of two adjacent boxes and the diameter of box. By Lemma 6.4 and Lemma 6.3, we obtain that

$$\begin{aligned} \text{diam}(B_{\mathbf{w}_v}^n) & \geq |C_1 \sigma^k \sigma^{2(n-k)} - C_2 \sigma^k \sigma^{n-k} b_1^{2^k}| \\ \text{dist}_{\min}(B_{\mathbf{w}_v}^n, B_{\mathbf{w}_c}^n) & \leq C_0 [\sigma^{2k} \sigma^{n-k} b_1^{2^k} + \sigma^{2k} \sigma^{2(n-k)} \bar{\varepsilon}^{2^k} + \sigma^k \sigma^{2(n-k)} b_2^{2^k}] \end{aligned}$$

where  $\mathbf{w} = v^k c v^{n-k-1} \in W^n$  for some numbers  $C_0 > 0$  and  $C_1$  and  $C_2$ . Hence,

$$\text{dist}_{\min}(B_{\mathbf{w}_v}^n, B_{\mathbf{w}_c}^n) \leq C \sigma^k \text{diam}(B_{\mathbf{w}_v}^n)$$

for every sufficiently large  $k \in \mathbb{N}$  and for some  $C > 0$ . Then the critical Cantor set has unbounded geometry.  $\square$

Overlapping is almost everywhere property in the sense of Lebesgue in [HLM]. See the following Theorem.

**Theorem 6.6** ([HLM]). *Given any  $0 < A_0 < A_1$ ,  $0 < \sigma < 1$  and any  $p \geq 2$ , the set of parameters  $b \in [0, 1]$  for which there are infinitely many  $0 < k < n$  satisfying*

$$A_0 < \frac{b^{p^k}}{\sigma^{n-k}} < A_1$$

*is a dense  $G_\delta$  set with full Lebesgue measure.*

Then unbounded geometry is almost everywhere property in the parameter set of  $b_1$  for every fixed  $b_2$ .

**Theorem 6.7.** *Let  $F_{b_1}$  be an element of parametrized space in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$  with  $b_1 = b_F/b_2$ . Then there exists a small interval  $[0, b_\bullet]$  for which there exists a  $G_\delta$  subset  $S \subset [0, b_\bullet]$  with full Lebesgue measure such that the critical Cantor set,  $\mathcal{O}_{F_{b_1}}$  has unbounded geometry for all  $b_1 \in S$ .*

## 7. Non rigidity on critical Cantor set

Let  $F$  and  $\tilde{F}$  be Hénon-like maps in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Let the universal number  $b_1$  and  $\tilde{b}_1$  are for the map  $F$  and  $\tilde{F}$  respectively. Non rigidity on critical Cantor set with respect to the universal constant  $b_1$  means that the homeomorphism between critical Cantor sets,  $\mathcal{O}_F$  and  $\mathcal{O}_{\tilde{F}}$  is at most  $\alpha$ -Hölder continuous with a constant  $\alpha < 1$  (Theorem 7.2 below). This kind of non rigidity phenomenon is a generalization of two dimensional one in [dCLM]. However, non rigidity of three dimension maps only depends essentially on the universal number,  $b_1$  from two dimensional Hénon-like map in three dimension.

**7.1. Bounds of the distance between two points.** Let us consider the box

$$B_{\mathbf{w}}^n = \Psi_0^k \circ F_k \circ \Psi_k^n(B)$$

where  $B = B(R^n F)$ . Since  $\text{diam } B(R^n F) \asymp \text{diam } B_v^1(R^n F)$ , by Lemma 6.4 we have the lower bound of  $\text{diam } B(R^n F)$  as follows

$$(7.1) \quad \text{diam}(B_{\mathbf{w}}^n) \geq |C_1 \sigma^k \sigma^{2(n-k)} - C_2 \sigma^k \sigma^{n-k} b_1^{2k}|$$

where  $\mathbf{w} = v^k c v^{n-k-1} \in W^n$  for some positive  $C_1$  and  $C_2$ .

**Lemma 7.1.** *Let  $F \in \mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Then*

$$\text{diam}(B_{\mathbf{w}}^n) \leq C [\sigma^k \sigma^{2(n-k)} + \sigma^k \sigma^{n-k} b_1^{2k}]$$

where  $\mathbf{w} = v^k c v^{n-k-1} \in W^n$  for some  $C > 0$ .

*Proof.* Recall the map  $\Psi_k^n$  from  $B(R^n F)$  to  $B_{\mathbf{v}}^n(R^k F)$ .

$$\Psi_k^n(w) = \begin{pmatrix} 1 & t_{n,k} & u_{n,k} \\ & 1 & \\ & d_{n,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,k} & & \\ & \sigma_{n,k} & \\ & & \sigma_{n,k} \end{pmatrix} \begin{pmatrix} x + S_k^n(w) \\ y \\ z + R_k^n(y) \end{pmatrix}.$$

where  $\mathbf{v} = v^{n-k} \in W^{n-k}$ . Let us choose two points

$$w_1 = (x_1, y_1, z_1) \in B_v^1(R^n F) \cap \mathcal{O}_{R^n F}, \quad w_2 = (x_2, y_2, z_2) \in B_c^1(R^n F) \cap \mathcal{O}_{R^n F}.$$

Recall  $\dot{w}_j = \Psi_k^n(w_j)$ ,  $\ddot{w}_j = F_k(\dot{w}_j)$  and  $\ddot{w}_j = \Psi_0^k(\ddot{w}_j)$  for  $j = 1, 2$ . Observe that  $|x_1 - x_2|$  and  $|y_1 - y_2|$  is  $O(1)$ . We may assume that  $|z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$  because  $\mathcal{O}_{R^n F}$  is a completely invariant set under  $R^n F$ . By Corollary 6.2 and the equation (6.5), we have

$$\dot{x}_1 - \dot{x}_2 = \alpha_{n,k} [(x_1 + S_k^n(w_1)) - (x_2 + S_k^n(w_2))]$$

$$\begin{aligned}
& + \sigma_{n,k} [t_{n,k}(y_1 - y_2) + u_{n,k} \{z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)\}] \\
& = \alpha_{n,k} [v'_*(\bar{x}) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k})] (x_1 - x_2) \\
& \quad + \sigma_{n,k} [t_{n,k}(y_1 - y_2) + u_{n,k} \{z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)\}] \\
(7.2) \quad & \leq C [\sigma^{2(n-k)} + \sigma^{n-k}(b_1^{2^k} + \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n})]
\end{aligned}$$

for some  $C > 0$ . Moreover,

$$\begin{aligned}
\dot{y}_1 - \dot{y}_2 &= \sigma_{n,k}(y_1 - y_2) \\
\dot{z}_1 - \dot{z}_2 &= \sigma_{n,k} [d_{n,k}(y_1 - y_2) + z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)].
\end{aligned}$$

By the equation (6.6), we estimate the distance between each coordinates of  $F_k(\dot{w}_1)$  and  $F_k(\dot{w}_2)$  as follows

$$\begin{aligned}
\ddot{x}_1 - \ddot{x}_2 &= f_k(\dot{x}_1) - \varepsilon_k(\dot{w}_1) - [f_k(\dot{x}_2) - \varepsilon_k(\dot{w}_2)] \\
&= f'_k(\bar{x}) \cdot (\dot{x}_1 - \dot{x}_2) - \varepsilon_k(\dot{w}_1) + \varepsilon_k(\dot{w}_2) \\
&= [f'_k(\bar{x}) - \partial_x \varepsilon_k(\eta)] \cdot (\dot{x}_1 - \dot{x}_2) - \partial_y \varepsilon_k(\eta) \cdot (\dot{y}_1 - \dot{y}_2) - \partial_z \varepsilon_k(\eta) \cdot (\dot{z}_1 - \dot{z}_2) \\
&= [f'_k(\bar{x}) - \partial_x \varepsilon_k(\eta)] \cdot (\dot{x}_1 - \dot{x}_2) - \partial_y \varepsilon_k(\eta) \cdot \sigma_{n,k}(y_1 - y_2) \\
&\quad - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k} [d_{n,k}(y_1 - y_2) + z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)] \\
\ddot{y}_1 - \ddot{y}_2 &= \dot{x}_1 - \dot{x}_2 \\
\ddot{z}_1 - \ddot{z}_2 &= \delta_k(\dot{w}_1) - \delta_k(\dot{w}_2) \\
&= \partial_x \delta_k(\zeta) \cdot (\dot{x}_1 - \dot{x}_2) + \partial_y \delta_k(\zeta) \cdot (\dot{y}_1 - \dot{y}_2) + \partial_z \delta_k(\zeta) \cdot (\dot{z}_1 - \dot{z}_2) \\
&= \partial_x \delta_k(\zeta) \cdot (\dot{x}_1 - \dot{x}_2) + \partial_y \delta_k(\zeta) \cdot \sigma_{n,k}(y_1 - y_2) \\
&\quad + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k} [d_{n,k}(y_1 - y_2) + z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)]
\end{aligned}$$

where  $\eta$  and  $\zeta$  are some points in the line segment between  $\dot{w}_1$  and  $\dot{w}_2$  in  $\Psi_k^n(B)$ . The equations (6.6) and (6.7) in Lemma 6.3 implies that

$$(7.3) \quad |\ddot{x}_1 - \ddot{x}_2| \leq |f'_k(\bar{x}) - \partial_x \varepsilon_k(\eta)| \cdot |\dot{x}_1 - \dot{x}_2| + C_2 \sigma^{n-k} [b_1^{2^k} + \bar{\varepsilon}^{2^k} \sigma^{n-k} + \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}]$$

and the equations (6.8) and (6.9) in the same Lemma implies that

$$(7.4) \quad |\ddot{z}_1 - \ddot{z}_2| \leq |\partial_x \delta_k(\zeta)| \cdot |\dot{x}_1 - \dot{x}_2| + C_3 \sigma^{n-k} [\sigma^{n-k} b_2^{2^k} + b_2^{2^k} \bar{\varepsilon}^{2^n}].$$

Then the difference of each coordinates of  $\Psi_0^k(\ddot{w}_1)$  and  $\Psi_0^k(\ddot{w}_2)$  as follows

$$\begin{aligned}
\ddot{x}_1 - \ddot{x}_2 &= \pi_x \circ \Psi_0^k(\ddot{w}_1) - \pi_x \circ \Psi_0^k(\ddot{w}_2) \\
&= \alpha_{k,0} [(\ddot{x}_1 + S_0^k(\ddot{w}_1)) - (\ddot{x}_2 + S_0^k(\ddot{w}_2))] \\
&\quad + \sigma_{k,0} [t_{k,0}(\ddot{y}_1 - \ddot{y}_2) + u_{k,0}(\ddot{z}_1 - \ddot{z}_2 + R_0^k(\ddot{y}_1) - R_0^k(\ddot{y}_2))]
\end{aligned}$$

$$(7.5) \quad \begin{aligned} &= \alpha_{k,0} [v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k)] (\ddot{x}_1 - \ddot{x}_2) + \sigma_{k,0} \cdot u_{k,0} (\ddot{z}_1 - \ddot{z}_2) \\ &\quad + \sigma_{k,0} [t_{k,0} (\dot{x}_1 - \dot{x}_2) + u_{k,0} (R_0^k(\dot{x}_1) - R_0^k(\dot{x}_2))] \end{aligned}$$

$$(7.6) \quad \ddot{y}_1 - \ddot{y}_2 = \sigma_{k,0} (\ddot{y}_1 - \ddot{y}_2) = \sigma_{k,0} (\dot{x}_1 - \dot{x}_2)$$

$$(7.7) \quad \begin{aligned} \ddot{z}_1 - \ddot{z}_2 &= \pi_z \circ \Psi_0^k(\ddot{w}_1) - \pi_z \circ \Psi_0^k(\ddot{w}_2) \\ &= \sigma_{k,0} (\ddot{z}_1 - \ddot{z}_2) + \sigma_{k,0} [d_{k,0}(\ddot{y}_1 - \ddot{y}_2) + R_k^n(\ddot{y}_1) - R_k^n(\ddot{y}_2)] \\ &= \sigma_{k,0} (\ddot{z}_1 - \ddot{z}_2) + \sigma_{k,0} [d_{k,0}(\dot{x}_1 - \dot{x}_2) + R_k^n(\dot{x}_1) - R_k^n(\dot{x}_2)]. \end{aligned}$$

Let us calculate a upper bound of the distance,  $|\ddot{w}_1 - \ddot{w}_2|$ . Applying the estimation (7.5), (7.6) and (7.7), we obtain that

$$\begin{aligned} |\ddot{w}_1 - \ddot{w}_2| &\leq |\ddot{x}_1 - \ddot{x}_2| + |\ddot{y}_1 - \ddot{y}_2| + |\ddot{z}_1 - \ddot{z}_2| \\ &\leq |\alpha_{k,0} [v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k)] (\ddot{x}_1 - \ddot{x}_2) + \sigma_{k,0} \cdot u_{k,0} (\ddot{z}_1 - \ddot{z}_2) \\ &\quad + \sigma_{k,0} [t_{k,0} (\dot{x}_1 - \dot{x}_2) + u_{k,0} (R_0^k(\dot{x}_1) - R_0^k(\dot{x}_2))] | \\ &\quad + |\sigma_{k,0} (\dot{x}_1 - \dot{x}_2)| + |\sigma_{k,0} (\ddot{z}_1 - \ddot{z}_2) + \sigma_{k,0} [d_{k,0}(\dot{x}_1 - \dot{x}_2) + R_k^n(\dot{x}_1) - R_k^n(\dot{x}_2)]| \\ &\leq |\alpha_{k,0} [v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k)] \cdot |f'_k(\bar{x}) - \partial_x \varepsilon_k(\eta)| \cdot |\dot{x}_1 - \dot{x}_2| \\ &\quad + |\alpha_{k,0} [v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k)] \cdot C_2 \sigma^{n-k} [b_1^{2k} + \bar{\varepsilon}^{2k} \sigma^{n-k} + \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n}]| \\ &\quad + |\sigma_{k,0} [1 + |t_{k,0}| + |d_{k,0}|] (\dot{x}_1 - \dot{x}_2)| + |\sigma_{k,0} [1 + |u_{k,0}|] \cdot |(R_k^n)'(\tilde{x}) \cdot (\dot{x}_1 - \dot{x}_2)| \\ &\quad + |\sigma_{k,0} [1 + |u_{k,0}|] \cdot [|\partial_x \delta_k(\zeta) \cdot (\dot{x}_1 - \dot{x}_2)| + C_3 \sigma^{n-k} (\sigma^{n-k} b_2^{2k} + b_2^{2k} \bar{\varepsilon}^{2n})] \end{aligned}$$

After factoring out  $|\dot{x}_1 - \dot{x}_2|$ , this inequality continues as follows

$$\begin{aligned} &\leq C_4 \sigma^k |\dot{x}_1 - \dot{x}_2| + C_5 \sigma^{2k} \sigma^{n-k} [b_1^{2k} + \bar{\varepsilon}^{2k} \sigma^{n-k} + \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n}] \\ &\quad + C_6 \sigma^k \sigma^{n-k} [\sigma^{n-k} b_2^{2k} + b_2^{2k} \bar{\varepsilon}^{2n}] \\ (*) &\leq C_7 \sigma^k [\sigma^{2(n-k)} + \sigma^{n-k} (b_1^{2k} + \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n})] \\ &\quad + C_5 \sigma^{2k} \sigma^{n-k} [b_1^{2k} + \bar{\varepsilon}^{2k} \sigma^{n-k} + \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n}] + C_6 \sigma^k \sigma^{n-k} [\sigma^{n-k} b_2^{2k} + b_2^{2k} \bar{\varepsilon}^{2n}] \end{aligned}$$

for some positive constants,  $C_j$ ,  $2 \leq j \leq 7$  which are independent of  $k$  and  $n$ . The second last line,  $(*)$  holds by the estimation (7.2) and  $\|(R_k^n)'\|$  at the end of Section 6.2. Observe that  $\sigma^n \gg \bar{\varepsilon}^{2n}$  for all big enough  $n$ . Then the above estimation continues

$$\begin{aligned} &\leq C_7 \sigma^k [\sigma^{2(n-k)} + \sigma^{n-k} (b_1^{2k} + \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n})] \\ &\quad + C_5 \sigma^{2k} \sigma^{n-k} [b_1^{2k} + \bar{\varepsilon}^{2k} \sigma^{n-k} + \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n}] + C_8 \sigma^k \sigma^{2(n-k)} b_2^{2k} \\ &\leq (C_7 + C_5 \sigma^k \bar{\varepsilon}^{2k} + C_6 b_2^{2k}) \sigma^k \sigma^{2(n-k)} + (C_7 + C_5 \sigma^k) \sigma^k \sigma^{n-k} b_1^{2k} \\ &\quad + (C_7 + C_5 \sigma^k) \sigma^k \sigma^{n-k} \bar{\varepsilon}^{2k} \bar{\varepsilon}^{2n} \end{aligned}$$



for some positive constant  $C_8$ . Moreover, if  $n \geq k + A$  where  $A$  is the number defined in Proposition 6.1, then

$$b_1^{2^k} \gtrsim \bar{\varepsilon}^{2^k} \bar{\varepsilon}^{2^n}.$$

Hence,

$$\text{diam}(B_{\mathbf{w}}^n) \leq C \left[ \sigma^k \sigma^{2(n-k)} + \sigma^k \sigma^{n-k} b_1^{2^k} \right]$$

where  $\mathbf{w} = v^k c v^{n-k-1} \in W^n$  for some  $C > 0$ .

□

*Remark 7.1.* Lemma 6.4 and Lemma 7.1 implies the lower and the upper bounds of  $\text{diam } B_{\mathbf{w}}$  where  $B_{\mathbf{w}} = \Psi_0^k \circ F_k \circ \Psi_k^n(B(R^n F))$  as follows

$$C_0 \sigma^k |\dot{x}_1 - \dot{x}_2| \leq \text{diam } B_{\mathbf{w}} \leq C_1 \sigma^k |\dot{x}_1 - \dot{x}_2|$$

for every big enough  $k \in \mathbb{N}$ , that is,  $\text{diam } B_{\mathbf{w}} \asymp \sigma^k |\dot{x}_1 - \dot{x}_2|$ .

**7.2. Non rigidity on critical Cantor set with respect to  $b_1$ .** Recall that  $b_1$  is  $b_F/b_2$  where  $b_F$  is the average Jacobian of  $F$  and  $b_2$  is the number defined in Proposition 4.3. The number  $\tilde{b}_1$  is defined by the similar way for the map  $\tilde{F}$ .

**Theorem 7.2.** *Let Hénon-like maps  $F$  and  $\tilde{F}$  be in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$ . Let  $\phi: \mathcal{O}_{\tilde{F}} \rightarrow \mathcal{O}_F$  be a homeomorphism which conjugate  $F_{\mathcal{O}_F}$  and  $\tilde{F}_{\mathcal{O}_{\tilde{F}}}$  and  $\phi(\tau_{\tilde{F}}) = \tau_F$ . If  $b_1 > \tilde{b}_1$ , then the Hölder*

*exponent of  $\phi$  is not greater than  $\frac{1}{2} \left( 1 + \frac{\log b_1}{\log \tilde{b}_1} \right)$ .*

*Proof.* Let two points  $w_1$  and  $w_2$  be in  $B_v^1(R^n F)$  and  $B_c^1(R^n F)$  respectively. Similarly, assume that  $\tilde{w}_1$  and  $\tilde{w}_2$  are in  $B(R^n \tilde{F})$ . Let us define  $\ddot{w}_j = \Psi_0^k \circ F_k \circ \Psi_k^n(w_j)$  for  $j = 1, 2$ . The points  $\ddot{\tilde{w}}_1$  and  $\ddot{\tilde{w}}_2$  are defined by the similar way. For sufficiently large  $k \in \mathbb{N}$ , let us choose  $n$  depending on  $k$  which satisfies the following inequality

$$\sigma^{n-k+1} \leq \tilde{b}_1^{2^k} < \sigma^{n-k}.$$

Observe that  $b_1^{2^k} \gg \tilde{b}_1^{2^k}$ . By Lemma 6.4 and Lemma 7.1, we have the following inequalities

$$\begin{aligned} \text{dist}(\ddot{\tilde{w}}_1, \ddot{\tilde{w}}_2) &\leq C_0 \left[ \sigma^k \sigma^{2(n-k)} + \sigma^k \sigma^{n-k} \tilde{b}_1^{2^k} \right] \leq C_1 \sigma^k \tilde{b}_1^{2^k} \tilde{b}_1^{2^k} \\ \text{dist}(\ddot{w}_1, \ddot{w}_2) &\geq |C_2 \sigma^k \sigma^{2(n-k)} - C_3 \sigma^k \sigma^{n-k} b_1^{2^k}| \geq C_4 \sigma^k \tilde{b}_1^{2^k} b_1^{2^k} \end{aligned}$$

for some positive  $C_j$  where  $j = 0, 1, 2, 3$  and 4. Hölder continuous function,  $\phi$  with the Hölder exponent  $\alpha$  has to satisfy

$$\text{dist}(\ddot{w}_1, \ddot{w}_2) \leq C \left( \text{dist}(\ddot{\tilde{w}}_1, \ddot{\tilde{w}}_2) \right)^\alpha$$

for some  $C > 0$ . Then we see that

$$\sigma^k \tilde{b}_1^{2^k} b_1^{2^k} \leq C \left( \sigma^k \tilde{b}_1^{2^k} \tilde{b}_1^{2^k} \right)^\alpha$$

Take the logarithm both sides and divide them by  $2^k$ . After passing the limit, divide both sides by the negative number,  $2 \log \tilde{b}_1$ . Then the desired upper bound of Hölder exponent is obtained

$$\begin{aligned} k \log \sigma + 2^k \log \tilde{b}_1 + 2^k \log b_1 &\leq \log C + \alpha (k \log \sigma + 2^k \log \tilde{b}_1 + 2^k \log \tilde{b}_1) \\ \frac{k}{2^k} \log \sigma + \log \tilde{b}_1 + \log b_1 &\leq \frac{1}{2^k} \log C + \alpha \left( \frac{k}{2^k} \log \sigma + \log \tilde{b}_1 + \log \tilde{b}_1 \right) \\ \log \tilde{b}_1 + \log b_1 &\leq \alpha \cdot 2 \log \tilde{b}_1 \\ \alpha &\leq \frac{1}{2} \left( 1 + \frac{\log b_1}{\log \tilde{b}_1} \right). \end{aligned}$$

□

In renormalization theory of two dimensional Hénon-like map, the answer of rigidity problem with average Jacobian is unknown. In other words, the best regularity which the conjugation  $\phi$  should satisfy is not known yet where  $b_F = b_{\tilde{F}}$  for two dimensional Hénon-like maps  $F$  and  $\tilde{F}$ . However, the average Jacobian of three dimensional Hénon-like map in  $\mathcal{N} \cap \mathcal{I}(\bar{\varepsilon})$  less affects rigidity than  $b_1$ . Moreover, in higher dimension, we do not expect rigidity with average Jacobian between Cantor attractors.

**Example 7.3.** Let us consider a map  $F$  in  $\mathcal{I}(\bar{\varepsilon})$  as follows

$$F(w) = (f(x) - \varepsilon(x, y), x, \delta(z)).$$

We call the three dimensional Hénon-like map satisfying  $\delta(w) \equiv \delta(z)$  a *trivial extension* of two dimensional Hénon-like map. Let the set of these maps be  $\mathcal{T}$ . If  $F \in \mathcal{T} \cap \mathcal{I}(\bar{\varepsilon})$ , then the  $n^{th}$  renormalized map of  $F$ ,  $F_n \equiv R^n F$  is as follows

$$F_n(w) = (f_n(x) - a(x) b_1^{2^n} y (1 + O(\rho^n)), x, b_2^{2^n} z (1 + O(\rho^n)))$$

where  $b_1$  is the average Jacobian of two dimensional map,  $\pi_{xy} \circ F$  and  $b_2 = b_F/b_1$  for some  $0 < \rho < 1$ . Let  $\tilde{F}$  be another map in  $\mathcal{T} \cap \mathcal{I}(\bar{\varepsilon})$  with the corresponding numbers  $\tilde{b}_1$ ,  $\tilde{b}$  and  $\tilde{b}_2$ . By Theorem 7.2, if  $b_1 > \tilde{b}_1$ , the upper bound of Hölder exponent is

$$\frac{1}{2} \left( 1 + \frac{\log b_1}{\log \tilde{b}_1} \right)$$

Let  $\delta$  and  $\tilde{\delta}$  be the third coordinate map of  $F$  and  $\tilde{F}$  respectively. Recall that  $b_1 b_2 = b_F$  for every map  $F \in \mathcal{T} \cap \mathcal{I}(\bar{\varepsilon})$  and  $b_2$  is the contracting rate along the third coordinate. The condition,  $b_2 \neq \tilde{b}_2$  may require non rigidity of homeomorphic conjugacy between critical Cantor sets of  $F$  and  $\tilde{F}$  even if  $b_F = b_{\tilde{F}}$ . Assume that

$$b_1 b_2 = b = \tilde{b} = \tilde{b}_1 \tilde{b}_2$$

Thus the condition  $b_2 \neq \tilde{b}_2$  implies either  $b_1 > \tilde{b}_1$  or  $b_1 < \tilde{b}_1$ . Then Theorem 7.2 implies the non rigidity between Cantor attractors of  $F$  and  $\tilde{F}$ .

## Appendix A

### Recursive formula of $\text{Jac } R^n F$

Recall the definition of  $H$  and  $H^{-1}$

$$\begin{aligned} H(x, y, z) &= (f(x) - \varepsilon(w), y, z - \delta(y, f^{-1}(y), 0)) \\ H^{-1}(x, y, z) &= (\phi^{-1}(w), y, z + \delta(y, f^{-1}(y), 0)). \end{aligned}$$

Thus  $\phi^{-1}(x, y, z)$  is the straightening map satisfying  $\phi^{-1} \circ H(w) = x$ .

$$f \circ \phi^{-1}(w) - \varepsilon \circ H^{-1}(w) = x.$$

Then

$$\phi^{-1}(w) = f^{-1}(x + \varepsilon \circ H(w))$$

Recall  $\varepsilon \circ H^{-1}(w) = \varepsilon(\phi^{-1}(w), y, z + \delta(y, f^{-1}(y), 0))$ . Then by the chain rule, each partial derivatives of  $\phi^{-1}$  is as follows

$$\begin{aligned} \partial_x \phi^{-1}(w) &= (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \cdot [1 + \partial_x \varepsilon \circ H^{-1}(w) \cdot \partial_x \phi^{-1}(w)] \\ \partial_y \phi^{-1}(w) &= (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \\ &\quad \cdot [\partial_x \varepsilon \circ H^{-1}(w) \cdot \partial_y \phi^{-1}(w) + \partial_y \varepsilon \circ H^{-1}(w) + \partial_z \varepsilon \circ H^{-1}(w) \cdot \frac{d}{dy} \delta(y, f^{-1}(y), 0)] \\ \partial_z \phi^{-1}(w) &= (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \cdot [\partial_x \varepsilon \circ H^{-1}(w) \cdot \partial_z \phi^{-1}(w) + \partial_z \varepsilon \circ H^{-1}(w)]. \end{aligned}$$

Then

$$\begin{aligned} \partial_x \phi^{-1}(w) &= \frac{(f^{-1})'(x + \varepsilon \circ H^{-1}(w))}{1 - (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \cdot \partial_x \varepsilon \circ H^{-1}(w)} \\ \partial_y \phi^{-1}(w) &= \partial_x \phi^{-1}(w) \cdot [\partial_y \varepsilon \circ H^{-1}(w) + \partial_z \varepsilon \circ H^{-1}(w) \cdot \frac{d}{dy} \delta(y, f^{-1}(y), 0)] \\ \partial_z \phi^{-1}(w) &= \partial_x \phi^{-1}(w) \cdot \partial_x \varepsilon \circ H^{-1}(w). \end{aligned}$$

Recall *pre-renormalization* of  $F$ ,  $PRF$  is defined as follows

$$PRF = H \circ F^2 \circ H^{-1}$$

where  $H(w) = (f(x) - \varepsilon(w), y, z - \delta(y, f^{-1}(y), 0))$ . Recall the renormalized map  $RF$  is defined as  $\Lambda \circ PRF \circ \Lambda^{-1}$  where  $\Lambda(w) = (sx, sy, sz)$  for the appropriate number  $s < -1$  from the renormalized one dimensional map,  $f(x)$ . Denote  $\sigma_0 = 1/s$ . Let the first coordinate map of  $H^{-1}(w)$  be  $\phi^{-1}(w)$ . Then

$$H^{-1}(w) = (\phi^{-1}(w), y, z + \delta(y, f^{-1}(y), 0)).$$

By the direct calculation  $PRF$  is as follows.

$$PRF(w) = (f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w), x, \delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0))$$

Let the perturbed part of the first coordinate map of  $PRF$  be  $\text{Pre } \varepsilon_1(w)$ . Let the third coordinate map of  $PRF$  be  $\text{Pre } \delta_1(w)$ . Moreover,  $\text{Pre } \varepsilon_1(w)$  and  $\text{Pre } \delta_1(w)$  is defined as the

corresponding parts of  $PR^kF$  for each  $k \in \mathbb{N}$ . Then relations between  $\text{Pre } \varepsilon_k(w)$  and  $\varepsilon_k(w)$  and between  $\text{Pre } \delta_k(w)$  and  $\delta_k(w)$  respectively are as follows

$$\text{Pre } \varepsilon_k(w) = \sigma_{k-1} \cdot \varepsilon_k \circ \left( \frac{w}{\sigma_{k-1}} \right) \quad \text{and} \quad \text{Pre } \delta_k(w) = \sigma_{k-1} \cdot \delta_k \circ \left( \frac{w}{\sigma_{k-1}} \right).$$

**Lemma A.1.** *Let  $F$  be a renormalizable three dimensional Hénon-like map. Let  $\delta_1$  be the third coordinate of  $RF$ , namely,  $\pi_z \circ RF$ . Then*

$$\begin{aligned} \partial_x \delta_1(w) &= \left[ \partial_y \delta \circ \psi_c^1(w) + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_x \delta \circ \psi_v^1(w) \right] \cdot \partial_x \phi^{-1}(\sigma_0 w) \\ &\quad + \partial_x \delta \circ \psi_c^1(w) - \frac{d}{dx} \delta(\sigma_0 x, f^{-1}(\sigma_0 x), 0) \\ \partial_y \delta_1(w) &= \left[ \partial_y \delta \circ \psi_c^1(w) + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_x \delta \circ \psi_v^1(w) \right] \cdot \partial_y \phi^{-1}(\sigma_0 w) \\ &\quad + \partial_z \delta \circ \psi_c^1(w) \cdot \left[ \partial_y \delta \circ \psi_v^1(w) + \partial_z \delta \circ \psi_v^1(w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right] \\ \partial_z \delta_1(w) &= \left[ \partial_y \delta \circ \psi_c^1(w) + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_x \delta \circ \psi_v^1(w) \right] \cdot \partial_z \phi^{-1}(\sigma_0 w) \\ &\quad + \partial_z \delta \circ \psi_c^1(w) \cdot \partial_z \delta \circ \psi_v^1(w) \end{aligned}$$

*Proof.* Let us calculate the recursive formula of each partial derivatives of  $\text{Pre } \delta_1(w)$ . Let us estimate  $\partial_x(\text{Pre } \delta_1(w))$ . Then

$$\begin{aligned} \partial_x (\delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0)) &= \frac{\partial}{\partial x} \delta(x, \phi^{-1}(x), \delta \circ H^{-1}(w)) - \frac{d}{dx} \delta(x, f^{-1}(x), 0) \\ &= \partial_x \delta \circ (F \circ H^{-1}(w)) + \partial_y \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \phi^{-1}(w) \\ &\quad + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x (\delta \circ H^{-1}(w)) - \frac{d}{dx} \delta(x, f^{-1}(x), 0) \\ &= \partial_x \delta \circ (F \circ H^{-1}(w)) + \partial_y \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \phi^{-1}(w) \\ &\quad + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w) \cdot \partial_x \phi^{-1}(w) - \frac{d}{dx} \delta(x, f^{-1}(x), 0) \\ &= \boxed{\left[ \partial_y \delta \circ (F \circ H^{-1}(w)) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w) \right]} \cdot \partial_x \phi^{-1}(w) \\ &\quad + \partial_x \delta \circ (F \circ H^{-1}(w)) - \frac{d}{dx} \delta(x, f^{-1}(x), 0). \end{aligned}$$

Let us estimate  $\partial_y(\text{Pre } \delta_1(w))$ . Then

$$\begin{aligned} \partial_y (\delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0)) &= \frac{\partial}{\partial y} \delta(x, \phi^{-1}(x), \delta \circ H^{-1}(w)) \\ &= \partial_y \delta \circ (F \circ H^{-1}(w)) \cdot \partial_y \phi^{-1}(w) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \frac{\partial}{\partial y} (\delta \circ H^{-1}(w)) \\ &= \partial_y \delta \circ (F \circ H^{-1}(w)) \cdot \partial_y \phi^{-1}(w) + \partial_z \delta \circ (F \circ H^{-1}(w)) \\ &\quad \cdot \left[ \partial_x \delta \circ H^{-1}(w) \cdot \partial_y \phi^{-1}(w) + \partial_y \delta \circ H^{-1}(w) + \partial_z \delta \circ H^{-1}(w) \cdot \frac{d}{dy} \delta(y, f^{-1}(y), 0) \right] \end{aligned}$$

$$\begin{aligned}
(A.1) &= \boxed{[\partial_y \delta \circ (F \circ H^{-1}(w)) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w)]} \cdot \partial_y \phi^{-1}(w) \\
&\quad + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \left[ \partial_y \delta \circ H^{-1}(w) + \partial_z \delta \circ H^{-1}(w) \cdot \frac{d}{dy} \delta(y, f^{-1}(y), 0) \right].
\end{aligned}$$

Similarly, we can estimate  $\partial_z(\text{Pre } \delta_1(w))$ . Then

$$\begin{aligned}
&\partial_z(\delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0)) = \frac{\partial}{\partial z} \delta(x, \phi^{-1}(x), \delta \circ H^{-1}(w)) \\
&= \partial_y \delta \circ (F \circ H^{-1}(w)) \cdot \partial_z \phi^{-1}(w) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \frac{\partial}{\partial z} (\delta \circ H^{-1}(w)) \\
&= \partial_y \delta \circ (F \circ H^{-1}(w)) \cdot \partial_z \phi^{-1}(w) \\
&\quad + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot [\partial_x \delta \circ H^{-1}(w) \cdot \partial_z \phi^{-1}(w) + \partial_z \delta \circ H^{-1}(w)] \\
(A.2) &= \boxed{[\partial_y \delta \circ (F \circ H^{-1}(w)) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w)]} \cdot \partial_z \phi^{-1}(w) \\
&\quad + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_z \delta \circ H^{-1}(w).
\end{aligned}$$

Since  $\psi_v^1(w) = H^{-1}(\sigma_0 w)$  and  $\psi_c^1(w) = F \circ H^{-1}(\sigma_0 w)$ , the fact that  $\partial_t \delta_1(w) = \partial_t(\text{Pre } \delta_1(\sigma_0 w))$  for  $t = x, y, z$  implies the equations in Lemma. The proof is complete.  $\square$

Let us estimate  $\partial_y(\text{Pre } \varepsilon_1(w))$ . Thus we need to estimate

$$\partial_y(\varepsilon \circ F \circ H^{-1}(w)) \text{ and } \partial_y(\varepsilon \circ F^2 \circ H^{-1}(w)).$$

firstly. Let us estimate  $\partial_y(\varepsilon \circ F \circ H^{-1}(w))$ . Then

$$\begin{aligned}
&\partial_y(\varepsilon \circ F \circ H^{-1}(w)) = \frac{\partial}{\partial y} \varepsilon(x, \phi^{-1}(x), \delta \circ H^{-1}(w)) \\
&= \partial_y \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_y \phi^{-1}(w) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \frac{\partial}{\partial y} (\delta \circ H^{-1}(w)) \\
&= \partial_y \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_y \phi^{-1}(w) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \\
&\quad \cdot \left[ \partial_x \delta \circ H^{-1}(w) \cdot \partial_y \phi^{-1}(w) + \partial_y \delta \circ H^{-1}(w) + \partial_z \delta \circ H^{-1}(w) \cdot \frac{d}{dy} \delta(y, f^{-1}(y), 0) \right] \\
&= [\partial_y \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w)] \cdot \partial_y \phi^{-1}(w) \\
&\quad + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \left[ \partial_y \delta \circ H^{-1}(w) + \partial_z \delta \circ H^{-1}(w) \cdot \frac{d}{dy} \delta(y, f^{-1}(y), 0) \right].
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\partial_y(\varepsilon \circ F^2 \circ H^{-1}(w)) = \frac{\partial}{\partial y} \varepsilon(f(x) - \varepsilon \circ F \circ H^{-1}(w), x, \delta \circ F \circ H^{-1}(w)) \\
&= -\partial_x \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \frac{\partial}{\partial y} (\varepsilon \circ F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \frac{\partial}{\partial y} (\delta \circ F \circ H^{-1}(w)).
\end{aligned}$$

The map  $f'(f_\varepsilon(x))$  denote the function  $f'(f(x) - \varepsilon \circ F \circ H^{-1}(w) - \partial_x \varepsilon \circ (F^2 \circ H^{-1}(w)))$ . Then  $\partial_y(\text{Pre } \varepsilon_1(w))$  can be expressed in terms of partial derivatives of  $\varepsilon(w)$  and  $\delta(w)$  as follows

$$\begin{aligned}
& \partial_y \text{Pre } \varepsilon_1(w) = -\partial_y [f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w)] \\
& = f'(f(x) - \varepsilon \circ F \circ H^{-1}(w)) \cdot \partial_y(\varepsilon \circ F \circ H^{-1}(w)) + \partial_y(\varepsilon \circ F^2 \circ H^{-1}(w)) \\
& = [f'(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \partial_x \varepsilon \circ (F^2 \circ H^{-1}(w))] \cdot \partial_y(\varepsilon \circ F \circ H^{-1}(w)) \\
& \quad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_y(\delta \circ F \circ H^{-1}(w)) \\
& \quad (A.3) \\
& = \left[ f'(f_\varepsilon(x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w) \} \right. \\
& \quad \left. + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \left\{ \boxed{\partial_y \delta \circ (F \circ H^{-1}(w)) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w)} \right\} \right] \\
& \quad \cdot \partial_y \phi^{-1}(w) \\
& + \left[ f'(f_\varepsilon(x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(w)) \right] \\
& \quad \cdot \left[ \partial_y \delta \circ H^{-1}(w) + \partial_z \delta \circ H^{-1}(w) \cdot \frac{d}{dy} \delta(y, f^{-1}(y), 0) \right].
\end{aligned}$$

Let us estimate  $\partial_z(\varepsilon \circ F \circ H^{-1}(w))$ . Then

$$\begin{aligned}
& \partial_z(\varepsilon \circ F \circ H^{-1}(w)) = \frac{\partial}{\partial z} \varepsilon(x, \phi^{-1}(x), \delta \circ H^{-1}(w)) \\
& = \partial_y \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_z \phi^{-1}(w) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \frac{\partial}{\partial z} (\delta \circ H^{-1}(w)) \\
& = \partial_y \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_z \phi^{-1}(w) \\
& \quad + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot [\partial_x \delta \circ H^{-1}(w) \cdot \partial_z \phi^{-1}(w) + \partial_z \delta \circ H^{-1}(w)] \\
& = [\partial_y \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w)] \cdot \partial_z \phi^{-1}(w) \\
& \quad + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_z \delta \circ H^{-1}(w).
\end{aligned}$$

Moreover, we can express  $\partial_y(\varepsilon \circ F^2 \circ H^{-1}(w))$  in terms of  $\partial_y(\varepsilon \circ F \circ H^{-1}(w))$  and  $\partial_y(\delta \circ F \circ H^{-1}(w))$  as follows

$$\begin{aligned}
& \partial_z(\varepsilon \circ F^2 \circ H^{-1}(w)) = \frac{\partial}{\partial z} \varepsilon(f(x) - \varepsilon \circ F \circ H^{-1}(w), x, \delta \circ F \circ H^{-1}(w)) \\
& = -\partial_x \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \frac{\partial}{\partial z} (\varepsilon \circ F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \frac{\partial}{\partial z} (\delta \circ F \circ H^{-1}(w)).
\end{aligned}$$

Then  $\partial_z(\text{Pre } \varepsilon_1(w))$  can be estimated in terms of partial derivatives of  $\varepsilon(w)$  and  $\delta(w)$

$$\begin{aligned}
& \partial_z \text{Pre } \varepsilon_1(w) = -\partial_z [f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w)] \\
& = f'(f(x) - \varepsilon \circ F \circ H^{-1}(w)) \cdot \partial_z(\varepsilon \circ F \circ H^{-1}(w)) + \partial_z(\varepsilon \circ F^2 \circ H^{-1}(w)) \\
& = [f'(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \partial_x \varepsilon \circ (F^2 \circ H^{-1}(w))] \cdot \partial_z(\varepsilon \circ F \circ H^{-1}(w)) \\
& \quad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_z(\delta \circ F \circ H^{-1}(w))
\end{aligned}$$

$$\begin{aligned}
& (A.4) \\
& = \left[ f'(f_\varepsilon(x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w) \} \right. \\
& \quad \left. + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \left\{ \partial_y \delta \circ (F \circ H^{-1}(w)) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w) \right\} \right] \\
& \quad \cdot \partial_z \phi^{-1}(w) \\
& + \left[ f'(f_\varepsilon(x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(w)) \right] \\
& \quad \cdot \partial_z \delta \circ H^{-1}(w).
\end{aligned}$$

**Lemma A.2.** *Let  $F$  be an infinitely renormalizable three dimensional Hénon-like map. Then*

$$\begin{aligned}
\text{Jac } R^n F(w) &= (f_{n-1}^{-1})'(\sigma_{n-1}x) \cdot f_{n-1}'(f_{n-1}(\sigma_{n-1}x)) \\
&\quad \cdot \text{Jac } R^{n-1} F \circ (H_{n-1}^{-1}(\sigma_{n-1}w)) \cdot \text{Jac } R^{n-1} F \circ (F_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1}w)).
\end{aligned}$$

*Proof.* Let us calculate  $\text{Jac } RF(w)$  in terms of partial derivatives of  $\varepsilon$  and  $\delta$ . Recall the equations (A.1), (A.2), (A.3) and (A.4). Let us express  $\text{Jac } RF$  in terms of these expressions

$$\begin{aligned}
\text{Jac } RF(w) &= \partial_y \varepsilon_1(w) \cdot \partial_z \delta_1(w) - \partial_z \varepsilon_1(w) \cdot \partial_y \delta_1(w) \\
&= \left[ \left\{ f'(f_\varepsilon(\sigma_0 x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \right. \right. \\
&\quad \left. + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \right\} \cdot \partial_y \phi^{-1}(\sigma_0 w) \\
&\quad + \left\{ f'(f_\varepsilon(\sigma_0 x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \right\} \\
&\quad \cdot \left\{ \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right\} \Big] \\
&\quad \cdot \left[ \left\{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \right\} \cdot \partial_z \phi^{-1}(\sigma_0 w) \right. \\
&\quad \left. + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \right] \\
&- \left[ \left\{ f'(f_\varepsilon(\sigma_0 x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \right. \right. \\
&\quad \left. + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \right\} \cdot \partial_z \phi^{-1}(\sigma_0 w) \\
&\quad + \left\{ f'(f_\varepsilon(\sigma_0 x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \right\} \\
&\quad \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \Big] \\
&\quad \cdot \left[ \left\{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \right\} \cdot \partial_y \phi^{-1}(\sigma_0 w) \right. \\
&\quad \left. + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \left\{ \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right\} \right]
\end{aligned}$$

On the above equation, let us denote some factors to be  $A$ ,  $B$ ,  $C$  and  $D$  as follows

$$\begin{aligned}
A &= f'(f_\varepsilon(\sigma_0 x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \\
&\quad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \\
&\quad \cdot \{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \\
B &= f'(f_\varepsilon(\sigma x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
C &= \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \\
D &= \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0).
\end{aligned}$$

Let us calculate  $A \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma w)) - BC$  for later use

$$\begin{aligned}
&A \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) - BC \\
&= \left[ f'(f_\varepsilon(\sigma_0 x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \right. \\
&\quad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \\
&\quad \cdot \{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \left. \right] \\
&\quad \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
&\quad - \left[ f'(f_\varepsilon(\sigma_0 x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \right] \\
&\quad \cdot \left[ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \right] \\
&\quad (A.5) \\
&= f'(f_\varepsilon(\sigma_0 x)) \cdot \left[ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \right. \\
&\quad \left. - \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) \right].
\end{aligned}$$

Then the above equation of  $\text{Jac } RF$  is expressed as follows

$$\begin{aligned}
&\partial_y \varepsilon_1(w) \cdot \partial_z \delta_1(w) - \partial_z \varepsilon_1(w) \cdot \partial_y \delta_1(w) \\
&= \left[ A \cdot \partial_y \phi^{-1}(\sigma_0 w) + BD \right] \cdot \left[ C \cdot \partial_z \phi^{-1}(\sigma_0 w) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \right] \\
&\quad - \left[ A \cdot \partial_z \phi^{-1}(\sigma_0 w) + B \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \right] \cdot \left[ C \cdot \partial_y \phi^{-1}(\sigma_0 w) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot D \right] \\
&= A \cdot \partial_y \phi^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) + BCD \cdot \partial_z \phi^{-1}(\sigma_0 w) \\
&\quad - \left[ AD \cdot \partial_z \phi^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) + BC \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \partial_y \phi^{-1}(\sigma_0 w) \right] \\
&= A \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \left\{ \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) + \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right\} \\
&\quad \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \\
&\quad + BC \cdot \left\{ \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right\}
\end{aligned}$$



$$\begin{aligned}
& \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \\
& - \left[ A \cdot \left\{ \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right\} \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \right. \\
& \quad \cdot \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) + BC \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \\
& \quad \left. \cdot \left\{ \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) + \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \right\} \right] \\
& = A \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
& \quad \cdot \left[ \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \right] \\
& \quad - BC \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \\
& \quad \cdot \left[ \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \right] \\
& = \left[ A \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) - BC \right] \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \\
& \quad \cdot \left[ \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \right].
\end{aligned}$$

By the equation (A.5), the above equation is continued as follows

$$\begin{aligned}
& = (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \left[ \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \right] \\
& \quad \cdot f'(f_\varepsilon(\sigma_0 x)) \cdot \left[ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \right. \\
& \quad \quad \left. - \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) \right] \\
& = f'(f_\varepsilon(\sigma_0 x)) \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \text{Jac } F \circ (H^{-1}(\sigma_0 w)) \cdot \text{Jac } F \circ (F \circ H^{-1}(\sigma_0 w)).
\end{aligned}$$

Similarly,  $\text{Jac } R^n F(w)$  is expressed in terms of the partial derivatives of  $\varepsilon_{n-1}$  and  $\delta_{n-1}$  by induction

$$\begin{aligned}
\text{Jac } R^n F(w) &= (f_{n-1, \varepsilon}^{-1})'(\sigma_{n-1} x) \cdot f'_{n-1}(f_{n-1, \varepsilon}(\sigma_{n-1} x)) \\
&\quad \cdot \text{Jac } F_{n-1} \circ (H_{n-1}^{-1}(\sigma_{n-1} w)) \cdot \text{Jac } F_{n-1} \circ (F_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1} w)).
\end{aligned}$$

□

## Appendix B

### $C^1$ conjugation of Hénon-like map $F \in \mathcal{N}$

**Lemma B.1.** *Let  $F$  and  $\tilde{F}$  be Hénon-like maps. Suppose that  $C^1$  diffeomorphism  $\Phi: B \rightarrow B$  is a conjugation between  $F$  and  $\tilde{F}$  where  $\Phi(w) = (x, y, \varphi(y, z))$ . Then  $F \in \mathcal{N}$  if and only if  $\tilde{F} \in \mathcal{N}$ .*

*Proof.* The coordinates of  $F$  and  $\tilde{F}$  are as follows

$$\begin{aligned}
(B.1) \quad F(x, y, z) &= (f(x) - \varepsilon(w), x, \delta(w)) \\
\tilde{F} &= (\tilde{f}(x) - \tilde{\varepsilon}(w), x, \tilde{\delta}(w)).
\end{aligned}$$

We may assume that  $\Phi \circ F = \tilde{F} \circ \Phi$ . Thus by chain rule

$$D\Phi \circ F(w) \cdot DF(w) = D\tilde{F} \circ \Phi(w) \cdot D\Phi(w).$$

Let us consider the map  $\varphi(w) = \varphi(x, y, z)$  rather than  $\varphi(y, z)$ . Then

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_x \varphi & \partial_y \varphi & \partial_z \varphi \end{pmatrix} \cdot \begin{pmatrix} f'(x) - \partial_x \varepsilon & -\partial_y \varepsilon & -\partial_z \varepsilon \\ 1 & 0 & 0 \\ \partial_x \delta & \partial_y \delta & \partial_z \delta \end{pmatrix} \\ &= \begin{pmatrix} \tilde{f}'(x) - \partial_x \tilde{\varepsilon} & -\partial_y \tilde{\varepsilon} & -\partial_z \tilde{\varepsilon} \\ 1 & 0 & 0 \\ \partial_x \tilde{\delta} & \partial_y \tilde{\delta} & \partial_z \tilde{\delta} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_x \varphi & \partial_y \varphi & \partial_z \varphi \end{pmatrix}. \end{aligned}$$

Then we obtain

$$\begin{aligned} & \partial_x \varphi \circ F(w) \cdot \{ f'(x) - \partial_x \varepsilon(w) \} + \partial_y \varphi \circ F(w) + \partial_z \varphi \circ F(w) \cdot \partial_x \delta(w) \\ &= \partial_x \tilde{\delta} \circ \Phi(w) + \partial_z \tilde{\delta} \circ \Phi(w) \cdot \partial_x \varphi(w) \\ & - \partial_x \varphi \circ F(w) \cdot \partial_y \varepsilon(w) + \partial_z \varphi \circ F(w) \cdot \partial_y \delta(w) = \partial_y \tilde{\delta} \circ \Phi(w) + \partial_z \tilde{\delta} \circ \Phi(w) \cdot \partial_y \varphi(w) \\ & - \partial_x \varphi \circ F(w) \cdot \partial_z \varepsilon(w) + \partial_z \varphi \circ F(w) \cdot \partial_z \delta(w) = \partial_z \tilde{\delta} \circ \Phi(w) \cdot \partial_z \varphi(w). \end{aligned}$$

Recall that  $F \in \mathcal{N}$  means that  $\partial_y \delta \circ F(w) + \partial_z \delta \circ F(w) \cdot \partial_x \delta(w) \equiv 0$ . Then

$$\begin{aligned} & [ \partial_y \tilde{\delta} \circ (\Phi \circ F)(w) + \partial_z \tilde{\delta} \circ (\Phi \circ F)(w) \cdot \partial_x \tilde{\delta} \circ \Phi(w) ] \cdot \partial_z \varphi \circ F(w) \cdot \partial_z \varphi(w) \\ &= \partial_z \varphi \circ F^2(w) \cdot \partial_z \varphi \circ F(w) \cdot \partial_z \varphi(w) \cdot [ \partial_y \delta \circ F(w) + \partial_z \delta \circ F(w) \cdot \partial_x \delta(w) ] \\ & - \partial_x \varphi \circ F^2(w) \cdot \partial_y \varepsilon \circ F(w) \cdot \partial_z \varphi \circ F(w) \cdot \partial_z \varphi(w) \\ & - \partial_x \varphi \circ F^2(w) \cdot \partial_z \varepsilon \circ F(w) \cdot \partial_x \varphi \circ F(w) \cdot \partial_z \varphi(w) \cdot \{ f'(x) - \partial_x \varepsilon(w) \} \\ & - \partial_x \varphi \circ F^2(w) \cdot \partial_z \varepsilon \circ F(w) \cdot \partial_z \varphi \circ F(w) \cdot \partial_z \varphi(w) \cdot \partial_x \delta(w) \\ & + \partial_x \varphi \circ F^2(w) \cdot \partial_z \varepsilon \circ F(w) \cdot [ - \partial_x \varphi \circ F(w) \cdot \partial_z \varepsilon(w) + \partial_z \varphi \circ F(w) \cdot \partial_z \delta(w) ] \cdot \partial_x \varphi(w) \\ & + \partial_z \varphi \circ F^2(w) \cdot \partial_z \delta \circ F(w) \cdot \partial_x \varphi \circ F(w) \cdot \partial_z \varphi(w) \cdot \{ f'(x) - \partial_x \varepsilon(w) \} \\ & - \partial_z \varphi \circ F^2(w) \cdot \partial_z \delta \circ F(w) \cdot [ - \partial_x \varphi \circ F(w) \cdot \partial_z \varepsilon(w) + \partial_z \varphi \circ F(w) \cdot \partial_z \delta(w) ] \cdot \partial_x \varphi(w). \end{aligned}$$

The detailed calculation to obtain above equation is left to the reader. The fact that  $\partial_x \varphi(w) \equiv 0$  implies that

$$\begin{aligned} & [ \partial_y \tilde{\delta} \circ (\Phi \circ F)(w) + \partial_z \tilde{\delta} \circ (\Phi \circ F)(w) \cdot \partial_x \tilde{\delta} \circ \Phi(w) ] \cdot \partial_z \varphi \circ F(w) \cdot \partial_z \varphi(w) \\ &= \partial_z \varphi \circ F^2(w) \cdot \partial_z \varphi \circ F(w) \cdot \partial_z \varphi(w) \cdot [ \partial_y \delta \circ F(w) + \partial_z \delta \circ F(w) \cdot \partial_x \delta(w) ]. \end{aligned}$$

Since the map  $\Phi(w) = (x, y, \varphi(y, z))$  is a diffeomorphism,  $\partial_z \varphi(y, z)$  is never zero for all  $w \in B$ . Thus we have the following equation

$$\begin{aligned}
& \partial_y \tilde{\delta} \circ (\Phi \circ F)(w) + \partial_z \tilde{\delta} \circ (\Phi \circ F)(w) \cdot \partial_x \tilde{\delta} \circ \Phi(w) \\
&= \partial_z \varphi \circ F^2(w) \cdot [\partial_y \delta \circ F(w) + \partial_z \delta \circ F(w) \cdot \partial_x \delta(w)].
\end{aligned}$$

Hence,  $F \in \mathcal{N}$  if and only if  $\tilde{F} \in \mathcal{N}$ .

□

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